MAJORATION DU NOMBRE DE VALEURS FRIABLES D'UN POLYNÔME

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The following is a translation of section 8 of [dlBD17], which is concerned with the following bound.

Theorem 1. *Let $x, M, N \geq 1$, $MN \leq x^2$, and $(a_m), (b_n)$ be two sequences bounded in absolute values by 1. Let $D \in \mathbb{Z}$ which is not a perfect square, and $V : \mathbb{R} \to \mathbb{C}$ be a smooth function with compact support inside $\mathbb{R}_+^*$. Then*

$$\sum_{M \leq m \leq 2M} \sum_{\substack{N \leq n \leq 2N \\ (n,m)=1}} a_m b_n \Big(\sum_{\substack{k \in \mathbb{N} \\ mn \mid k^2 - D}} V\Big(\frac{k}{x}\Big) - x\widehat{V}(0)\frac{\varrho(mn)}{mn} \Big) \ll_{D,V} x^{\frac{1}{2}+\varepsilon} M^{\frac{1}{2}} + x^{1+\varepsilon} N^{\frac{3}{2}-\theta} M^{-\frac{1}{4}+\theta/2}, \tag{1}$$

where $\widehat{V}(\xi) = \int_{\mathbb{R}} V(t) \mathrm{e}(-t\xi) \mathrm{d}t$.

Another side-result is the following version of the main theorem of [DI82], with explicit dependence on the best-known bound towards Selberg's eigenvalue conjecture. Define

$$P_D(x) = P^+\Big(\prod_{x < n \leq 2x} (n^2 - D) \Big).$$

Corollary 2. *For $\theta \in [0, 1/4]$, let $\kappa(\theta) \in [1, 2]$ be the unique number satisfying*

$$\int_1^{\kappa(\theta)} \frac{t \mathrm{d}t}{1 - 2\theta t} = \frac{1}{4(1-2\theta)}.$$

For all $\varepsilon > 0$ and $D \in \mathbb{Z}$ which is not a perfect square, we have

$$P_D(x) \gg_{\varepsilon,D} x^{\kappa(\theta)-\varepsilon}$$

for all $\theta \geq 0$ which is admissible for Selberg's eigenvalue conjecture. In particular, $\theta = 7/64$ is admissible [Kim03]*; therefore, for x large enough,*

$$P_D(x) \geq x^{1{,}2182}.$$

Theorem 1 follows immediately from the following bound, using Cauchy–Schwarz's inequality.

Proposition 3. *Let $\varepsilon > 0$, $x, M, N \geq 1$, $MN \leq x^2$, $(b_n) \in \mathbb{C}^{\mathbb{N}}$ with $\|b\|_\infty \leq 1$, $D \in \mathbb{Z}$ which is not a perfect square, and $V : \mathbb{R} \to \mathbb{C}$ a smooth function compactly supported inside $\mathbb{R}_+^*$. Then*

$$\sum_{M < m \leq 2M} \Big| \sum_{\substack{N < n \leq 2N \\ (n,m)=1}} b_n \Big(\sum_{\substack{k \in \mathbb{N} \\ mn \mid k^2 - D}} V\Big(\frac{k}{x}\Big) - x\widehat{V}(0)\frac{\varrho(mn)}{mn} \Big) \Big|^2 \ll_{\varepsilon,V,D} \Big(1 + x \Big(\frac{M}{N^2}\Big)^{-\frac{3}{2}+\theta} \Big) x^{1+\varepsilon}. \tag{2}$$

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Remark. The left-hand side of (2) is trivially bounded by $M^{-1}x^{2+\varepsilon}$, which allows us to assume from the start that $M \geq N^2$, and justify the interest of seeking a value of θ as small as possible.

The previous proposition will be deduced from the following lemma, which is concerned with equidistribution of roots of quadratic congruences.

Lemma 4. *Let $(q,r,d)\in\mathbb{N}^3$ with $(q,2Dr)=1$ and $d|q$, $\lambda \pmod d$ an invertible class, and $\omega \pmod d$ a residue class such that $\omega^2\equiv D \pmod d$. Let $M \gg qd$, f be a smooth function compactly supported in $\mathbb{R}_+^*$, satisfying*

$$\|f^{(j)}\|_\infty \ll_j 1, \tag{3}$$

Let $0\leq\alpha<\beta<1$ and

$$P_f(M;q,r,d,\lambda,\omega,\alpha,\beta) := \sum_{\substack{(m,\Omega)\in\mathcal{D}\\ \alpha\leq\frac{\Omega}{mq}<\beta}} f\Big(\frac{m}{M}\Big), \tag{4}$$

where $\mathcal{D}$ is the set of pairs (m,Ω) such that

$$(m,qr)=1,\quad m\equiv\lambda \pmod d$$
$$\Omega^2\equiv D \pmod{mq},\quad \Omega\equiv\omega \pmod d.$$

Then for all $\varepsilon>0$, we have

$$P_f(M;q,r,d,\lambda,\omega,\alpha,\beta) = A_f(M;q,r,d,\alpha,\beta) + O_{\varepsilon,D,f}\Big((qrM)^\varepsilon d^{\frac34}(qd)^{\frac12-\theta}M^{\frac12+\theta}\Big). \tag{5}$$

Here, the main term A_f is defined through

$$A_f(M;q,r,d,\alpha,\beta) = (\beta-\alpha)M\widehat{f}(0)C_D\frac{A(qr)\rho(q/(q,d^\infty))}{\varphi(d)},$$

where $A(qr)=\prod_{p|qr}(1+1/p)^{-1}$ and $C_D>0$ is a constant depending only on D. The implicit constant depends at most on ε, D, and on the implicit constants in (3).

The previous estimate follows from the following exponential sums bound.

Lemma 5. *Under the notations and hypotheses of Lemma 4, The following bounds hold for all $\varepsilon>0$.*

(1) For $1\leq|h|\leq qd^{\frac12}$,

$$\sum_{(m,\Omega)\in\mathcal{D}} f\Big(\frac{m}{M}\Big)\mathrm{e}\Big(\frac{h\Omega}{mq}\Big) \ll_{\varepsilon,D,f} |h|(qr)^\varepsilon + (rM)^\varepsilon d^{\frac34}(qd,h)^\theta(qd)^{\frac12-\theta}M^{\frac12+\theta}. \tag{6}$$

(2) For $\frac12\leq H\ll qM$,

$$\frac1H\sum_{H<|h|\leq 2H}\Big|\sum_{(m,\Omega)\in\mathcal{D}} f\Big(\frac{m}{M}\Big)\mathrm{e}\Big(\frac{h\Omega}{mq}\Big)\Big| \ll_{\varepsilon,D,f} H(qr)^\varepsilon + (rM)^\varepsilon d^{\frac34}(qd)^{\frac12-\theta}M^{\frac12+\theta}. \tag{7}$$

(3) Assume $d=1$, $\frac12\leq Q\ll M$, $\frac12\leq H\ll QM$, that $t\in[0,1]$, let $\mathcal{I}\subset[H,2H]$ be an interval, and $(f_q)_{Q<q\leq 2Q}$ a sequence of functions satisfying (3) *and $f_q(v)\neq 0\Rightarrow v\asymp 1$ uniformly in q. Then*

$$\frac1Q\sum_{\substack{Q<q\leq 2Q\\(q,2Dr)=1}}\Big|\frac1H\sum_{h\in\mathcal{I}}\mathrm{e}(th)\sum_{(m,\Omega)\in\mathcal{D}} f_q\Big(\frac{m}{M}\Big)\mathrm{e}\Big(\frac{h\Omega}{mq}\Big)\Big| \ll_{\varepsilon,D,f} H(Qr)^\varepsilon + (rM)^\varepsilon\Big\{M^{\frac12}+H^{-\frac12}Q^{\frac12-\theta}M^{\frac12+\theta}\Big\}. \tag{8}$$

The implied constants depend at most on ε, D, and on the implicit constants in (3).

Remarks.

- When $d = 1$, using the Selberg bound $\theta \leq 1/4$ (*cf.* [DI83], theorem 4), we recover, up to the uniformity in h, a result of Duke, Friedlander and Iwaniec [DFI95, formula (25)] and Tóth [Tót00, formula (15)].
- The bounds (6) and (7) are valid for $M \ll qd$, but they are then less precise than the trivial bound $O_{\varepsilon,D,f}(q^{\varepsilon} M^{1+\varepsilon})$.

0.1. **Proof of Lemma 5.** We focus first on the bound (6). We assume that $h > 0$, taking complex conjugates if necessary.

0.1.1. *Coprimality.* Let $S'(M, q, \lambda)$ be the LHS of (6). A MÃűbius inversion yields

$$S'(M, q, \lambda) = \sum_{\substack{\ell | qr \\ (\ell, d)=1}} \mu(\ell) S(M/\ell, q\ell, \lambda\bar{\ell}) \tag{9}$$

where μ is the MÃűbius function and

$$S(M, q, \lambda) := \sum_{\substack{m \in \mathbb{N} \\ m \equiv \lambda \pmod{d}}} \sum_{\substack{\Omega \in \mathbb{N} \\ \alpha mq \leq \Omega < \beta mq \\ \Omega^2 \equiv D \pmod{mq} \\ \Omega \equiv \omega \pmod{d}}} f\Big(\frac{m}{M}\Big) \mathrm{e}\Big(\frac{h\Omega}{mq}\Big).$$

It will suffice to show that $S(M, q, \lambda)$ is bounded by the RHS of (6).

0.1.2. *Gauss correspondance.* Let

$$\mathscr{Q}_D = \{Q(X, Y) = AX^2 + 2BXY + CY^2,\ (A, B, C) \in \mathbb{Z}^3,\ B^2 - AC = D\}.$$

For $Q \in \mathscr{Q}_D$, we let $(A(Q), B(Q), C(Q))$ be the coefficients in the expression above. The group $\Gamma = PSL_2(\mathbb{Z})$ acts on $\mathscr{Q}_D$ through

$$\sigma Q(x, y) = Q((x, y)\sigma) \qquad (\sigma \in \Gamma)$$

where the product on the RHS is the matrix product. In particular,

$$\begin{aligned} B(\sigma Q) &= \alpha\gamma A + (\alpha\delta + \beta\gamma)B + \beta\delta C, \\ C(\sigma Q) &= \gamma^2 A + 2\gamma\delta B + \delta^2 C = Q(\gamma, \delta) \end{aligned} \tag{10}$$

if $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. By a reasonning identical to [DFI95, p. 247] (see also [Kow04, section 6.1]), we obtain

$$S(M, q, \lambda) = \sum_{Q \in \Gamma \backslash \mathscr{Q}_D} \sum_{\substack{\sigma \in \Gamma_\infty \backslash \Gamma / \Gamma_Q \\ \mathscr{P}(\sigma)}} f\Big(\frac{C(\sigma Q)}{qM}\Big) \mathrm{e}\Big(\frac{hB(\sigma Q)}{C(\sigma Q)}\Big), \tag{11}$$

where $\mathscr{P}(\sigma)$ is the property that

$$\mathscr{P}(\sigma) \Leftrightarrow \begin{cases} C(\sigma Q) \equiv \lambda q \pmod{qd}, \\ B(\sigma Q) \equiv \omega \pmod{d}, \end{cases}$$

and $\Gamma_Q \subset \Gamma$ is the stabilizer of Q.

0.1.3. *Localization of variables.* Let $\sigma = \left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right)$ be a generic element in index of the sum in the RHS of (11). We introduce, following Tóth [Tót00, lemme 4.2], a function $\Psi : \Gamma \to \mathbb{R}$ which allows us to encode the quotient by Γ_Q. This function satisfying $\sum_{\tau\in\Gamma_Q}\Psi(\sigma\tau) = 1$ for all $\sigma \in \Gamma$. When $D < 0$, the function Ψ is constant, and in the opposite case $\Psi(\sigma)$ is a $\mathcal{C}^\infty$ function of the ratio δ/γ. We also let $w : \mathbb{R} \to \mathbb{R}_+$ be a smooth function satisfying

$$\mathbf{1}_{|t|\leq\frac12} \leq w(t) \leq \mathbf{1}_{|t|\leq 2}, \qquad w(t) + w(1/t) = 1$$

for $t \neq 0$. We insert the weight $w(\gamma/\delta) + w(\delta/\gamma)$ in the RHS of (11). In the contribution of the term $w(\gamma/\delta)$, we apply to the sums over σ and Q the involutions changing $Q(X,Y)$ to $\widetilde{Q} = Q(Y,X)$, and σ to $\widetilde{\sigma} = \left(\begin{smallmatrix} -\beta & -\alpha \\ \delta & \gamma \end{smallmatrix}\right)$. We then have

$$B(\widetilde{\sigma}\widetilde{Q}) = -B(\sigma Q), \qquad C(\widetilde{\sigma}\widetilde{Q}) = C(\sigma Q).$$

We obtain

$$S(M,q,\lambda) = S(h,\Psi_1) + S(-h,\Psi_2), \tag{12}$$

with

$$(\Psi_1(\sigma),\Psi_2(\sigma)) = (w(t)\Psi(t), w(t)\Psi(1/t)) \qquad (\sigma = \left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_\infty\backslash\Gamma, \quad t = \gamma/\delta), \tag{13}$$

$$F_{\Psi,Q}(\sigma) = \Psi(\sigma) f\Big(\frac{C(\sigma Q)}{qM}\Big), \tag{14}$$

and

$$S(h,\Psi) = \sum_{Q\in\Gamma\backslash\mathscr{Q}_D} \sum_{\substack{\sigma\in\Gamma_\infty\backslash\Gamma/\Gamma_Q \\ \mathscr{P}(\sigma)}} F_{\Psi,Q}(\sigma) \mathrm{e}\Big(\frac{hB(\sigma Q)}{C(\sigma Q)}\Big). \tag{15}$$

In what follows, we let $\Psi \in \{\Psi_1, \Psi_2\}$ be fixed, noting that this function (and so the associated function $F_{\Psi,Q}$) vanishes whenever $|t| \geq 2$ (with the notation (13)).

0.1.4. *Simplification of the phase.* We write the definition (15) as $S(h,\Psi) = \sum_{Q\in\Gamma\backslash\mathscr{Q}_D} S_Q(h,\Psi)$. The sum over Q is finite and its number of terms depends at most on D. It will therefore suffice to bound $S_Q(h,\Psi)$ separately for each Q. For $\sigma \in \Gamma$, we define

$$\phi_\sigma = \frac{\alpha}{\gamma} \in \mathbb{R}/\mathbb{Z} \qquad (\sigma = \left(\begin{smallmatrix} \alpha & * \\ \gamma & * \end{smallmatrix}\right), \quad \gamma \neq 0).$$

The identity, due to Hooley [Hoo63, formula (27)],

$$\mathrm{e}\Big(\frac{hB(\sigma Q)}{C(\sigma Q)}\Big) = \mathrm{e}(h\phi_\sigma) + O(h(qM)^{-1}) \tag{16}$$

is then established similarly to lemma 4.3 of Toth [Tót00]. Inserting this in $S_Q(h,\Psi)$, we obtain

$$S_Q(h,\Psi) = T_Q(h, F_{\Psi,Q}) + O(h), \tag{17}$$

with

$$T_Q(h,F) = \sum_{\substack{\sigma\in\Gamma_\infty\backslash\Gamma/\Gamma_Q \\ \mathscr{P}(\sigma)}} F(\sigma)\mathrm{e}(h\phi_\sigma).$$

0.1.5. *Congruence conditions.* We decompose by the Hecke congruence subgroup $\Gamma_0(qd)$ to obtain

$$T_Q(h,\Psi) = \sum_{\substack{\tau\in\Gamma_\infty\backslash\Gamma_0(qd)\\ \sigma\in\Gamma_0(qd)\backslash\Gamma \\ \mathscr{P}(\tau\sigma)}}\sum F(\tau\sigma)\mathrm{e}(h\phi_{\tau\sigma}).$$

If $\tau = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_0(qd)$, then the relations (10) as well as $qd|\gamma$ show that

$$\mathscr{P}(\tau\sigma) \Leftrightarrow \begin{cases} q \mid C(\sigma Q), \\ \delta^2 q^{-1} C(\sigma Q) \equiv \lambda \pmod{d}, \\ B(\sigma Q) \equiv \omega \pmod{d}. \end{cases}$$

Since $(\lambda, d) = 1$, the second condition is detected by Dirichlet character, which yields

$$T_Q(h,\Psi) = \frac{1}{\varphi(d)} \sum_{\chi \pmod{d}} \overline{\chi(\lambda)} \sum_{\substack{\sigma\in\Gamma_0(qd)\backslash\Gamma \\ \mathscr{P}^*(\sigma)}} \chi(q^{-1}C(\sigma Q)) U_{Q,\sigma}(h,\Psi), \tag{18}$$

with

$$U(h,\Psi) = U_{Q,\sigma}(h,\Psi) = \sum_{\tau\in\Gamma_\infty\backslash\Gamma_0(qd)} \overline{\vartheta(\tau)} F(\tau\sigma)\mathrm{e}(h\phi_{\tau\sigma}),$$

where $\mathscr{P}^*(\sigma)$ now denotes the conditions

$$\mathscr{P}^*(\sigma) \Leftrightarrow \begin{cases} q \mid C(\sigma Q) \\ B(\sigma Q) \equiv \omega \pmod{d} \end{cases} \tag{19}$$

and ϑ denotes the central character defined by

$$\vartheta(\tau) = \overline{\chi}^2(\delta) \qquad (\tau = \left(\begin{smallmatrix} * & * \\ * & \delta\end{smallmatrix}\right) \in \Gamma_0(qd)).$$

0.1.6. *Reminders on generalized Kloosterman sums.* In this section, we recall some facts on Kloosterman sums. We refer to chapters 2 and 4 of [Iwa97] for definitions. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{P}^1(\mathbb{R})$ be two cusps for $\Gamma_0(qd)$, of stabilizers $\Gamma_\mathfrak{a}$ and $\Gamma_\mathfrak{b}$ and scaling matrices $\sigma_\mathfrak{a}, \sigma_\mathfrak{b} \in PSL_2(\mathbb{R})$, meaning that

$$\Gamma_\mathfrak{a} = \sigma_\mathfrak{a}\Gamma_\infty\sigma_\mathfrak{a}^{-1}, \qquad \Gamma_\mathfrak{b} = \sigma_\mathfrak{b}\Gamma_\infty\sigma_\mathfrak{b}^{-1}.$$

A cusp is equivalent under the action of $\Gamma_0(qd)$ to a unique cusp $\mathfrak{a}' = u/v$, with

$$v \geq 1, \quad v|qd, \quad (u,v) = 1, \quad 1 \leq u \leq (v, qd/v).$$

We may then define the *width* of the cusp $\mathfrak{a}$ to be the number

$$w_\mathfrak{a} = \frac{q}{(q, v^2)}. \tag{20}$$

We associate to $(\mathfrak{a}, \mathfrak{b})$ the set of moduli

$$\mathcal{C}(\mathfrak{a},\mathfrak{b}) := \left\{ c \in \mathbb{R}_+^* :\ \exists \alpha,\beta,\delta \in \mathbb{R}, \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{b} \right\}.$$

For all $c \in \mathcal{C}(\mathfrak{a},\mathfrak{b})$ et $(m,n) \in \mathbb{Z}^2$, we define the Kloosterman sums

$$S_{\mathfrak{a}\mathfrak{b}}(m,n;\gamma) = \sum_{\substack{\delta\in[0,\gamma)\ : \\ \left(\begin{smallmatrix}\alpha & * \\ \gamma & \delta\end{smallmatrix}\right) \in \sigma_\mathfrak{a}^{-1}\Gamma_0(qd)\sigma_\mathfrak{b}}} \overline{\vartheta}(\sigma_\mathfrak{a}\left(\begin{smallmatrix}\alpha & * \\ \gamma & \delta\end{smallmatrix}\right)\sigma_\mathfrak{b}^{-1}) \mathrm{e}\Big(\frac{\alpha m + \delta n}{\gamma}\Big). \tag{21}$$

We refer to section 4.1.1 of [Dra17] for more details, notably on the dependence of $S_{\mathfrak{a}\mathfrak{b}}(m,n;c)$ with respect to the scaling matrices $(\sigma_\mathfrak{a}, \sigma_\mathfrak{b})$. In this work, we will use the following facts.

Lemma 6. *Let $\sigma \in \Gamma_0(qd)\backslash\Gamma$ satisfy the conditions $\mathscr{P}^*(\sigma)$ defined in* (19).

(1) The number of such σ is $O(d\tau(q))$.

(2) Suppose that the cusp $\mathfrak{a} = \sigma\infty$ is equivalent to u/v, with $v|q$, $1 \leq u < v$ and $(u,v)=1$. Then $v|Q(0,1)^2$, in particular $v = O_Q(1)$, and

$$w_{\mathfrak{a}} = \frac{qd}{(qd,v^2)} \asymp_Q qd. \tag{22}$$

(3) The set of moduli $\mathcal{C}(\infty,\mathfrak{a})$ is

$$\mathcal{C}(\infty,\mathfrak{a}) = \Big\{ w_{\mathfrak{a}}^{\frac12} vm,\ m \in \mathbb{Z}:\ (m, qd/v) = 1 \Big\}.$$

(4) When $\gamma = w_{\mathfrak{a}}^{\frac12} vm \in \mathcal{C}(\infty,\mathfrak{a})$, the Kloosterman sum $S_{\infty\mathfrak{a}}(h,n;\gamma)$ is given by

$$S_{\infty\mathfrak{a}}(h,n;\gamma) = \sum_{\substack{\alpha \pmod{vm} \\ \delta \pmod{u[v,v']m} \\ \delta\equiv m \pmod{uv'} \\ (\delta-m,um)=u \\ \alpha\delta\equiv u \pmod{vm}}} \overline{\vartheta}\left(\begin{smallmatrix} * & * \\ * & \delta \end{smallmatrix}\right) \mathrm{e}\Big(\frac{h\alpha}{vm} + \frac{n\delta}{u[v,v']m}\Big),$$

where we have put $v' = qd/v$. Here, the scaling matrices are

$$\sigma_\infty = \mathrm{Id}, \qquad \sigma_{\mathfrak{a}} = \begin{pmatrix} u\sqrt{w_{\mathfrak{a}}} & 0 \\ v\sqrt{w} & (u\sqrt{w})^{-1} \end{pmatrix}.$$

(5) We have the trivial bound

$$|S_{\infty\mathfrak{a}}(h,n;\gamma)| \leq \frac{v}{(v,v')}(m,u)m \ll_Q m, \tag{23}$$

(6) When $n=0$, we have

$$|S_{\infty\mathfrak{a}}(h,0;\gamma)| \leq \tau(2m)^{O_{\mathfrak{a},Q}(1)}(dh,m). \tag{24}$$

The proof of this lemma, which is independant from the rest of the proof of Lemma 5, will be given below in section 0.5.

0.1.7. *Completion of sums.* In the sum $U(h,\Psi)$, we change τ to $\tau\sigma^{-1}$, so that

$$U(h,\Psi) = \sum_{\tau\in\Gamma_\infty\backslash\Gamma_0(qd)\sigma} \vartheta(\tau\sigma^{-1})F(\tau)\mathrm{e}(h\phi_\tau).$$

The cusp $\mathfrak{a} = \sigma\infty$ is equivalent to u/v for some $v|qd$ and $(u,v)=1$, and this expression is unique if we impose $1 \leq u \leq (v, qd/v)$. We temporarily write

$$\tau_{\mathfrak{a}} = \begin{pmatrix} w_{\mathfrak{a}}^{1/2} & 0 \\ 0 & w_{\mathfrak{a}}^{-1/2} \end{pmatrix}, \qquad \sigma_{\mathfrak{a}} = \sigma\tau_{\mathfrak{a}}$$

so that the stabilize $\Gamma_{\mathfrak{a}} \subset \Gamma_0(qd)$ of $\mathfrak{a}$ satisfies $\Gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}$. In the sum on the RHS of (25), we replace again τ by $\tau\tau_{\mathfrak{a}}^{-1}$ noting that this leaves the quantity ϕ_τ unchanged. We obtain

$$U(h,\Psi) = \sum_{\tau\in\Gamma_\infty\backslash\Gamma_0(qd)\sigma_{\mathfrak{a}}} \vartheta(\tau\sigma_{\mathfrak{a}}^{-1})F(\tau\tau_{\mathfrak{a}}^{-1})\mathrm{e}(h\phi_\tau). \tag{25}$$

At this point, we remark that $(v, qd/v)|q$. In particular, the cusp $\mathfrak{a}$ is singular for ϑ, meaning that

$$\vartheta(\tau) = 1 \qquad (\tau \in \Gamma_{\mathfrak{a}}).$$

We separate the sum over τ in the RHS of (25) according to right classes modulo Γ_∞. Note that for $\omega \in \Gamma_\infty$, we have $\phi_{\tau\omega} \equiv \phi_\tau \pmod 1$, as well as

$$\vartheta(\tau\omega\sigma_{\mathfrak{a}}^{-1}) = \vartheta(\tau\sigma_{\mathfrak{a}}^{-1})\vartheta(\sigma_{\mathfrak{a}}\omega\sigma_{\mathfrak{a}}^{-1}) = \vartheta(\tau\sigma^{-1}).$$

We obtain

$$U(h,\Psi) = \sum_{\tau\in\Gamma_\infty\backslash\Gamma_0(qd)\sigma_{\mathfrak{a}}/\Gamma_\infty} \vartheta(\tau\sigma_{\mathfrak{a}}^{-1})\mathrm{e}(h\phi_\tau)\sum_{k\in\mathbb{Z}} F\left(\tau\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)\tau_{\mathfrak{a}}^{-1}\right).$$

Given the relations (13) and (14), the function

$$t \mapsto F\Big(\tau\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\tau_{\mathfrak{a}}^{-1}\Big)$$

is smooth, with compact support, and only depends on the lower entries of τ. If $\tau = \left(\begin{smallmatrix}* & *\\ \gamma & \delta\end{smallmatrix}\right)$, then

$$F\Big(\tau\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\tau_{\mathfrak{a}}^{-1}\Big) = F\Big(\begin{pmatrix}* & *\\ \gamma w_{\mathfrak{a}}^{-1/2} & \gamma(t+\delta/\gamma)w_{\mathfrak{a}}^{1/2}\end{pmatrix}\Big).$$

The Poisson summation formula yields

$$\sum_{k\in\mathbb{Z}} F\Big(\tau\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}\tau_{\mathfrak{a}}^{-1}\Big) = \mathrm{e}\Big(\frac{n\delta}{\gamma}\Big)\sum_{n\in\mathbb{N}} G(\gamma,n),$$

where

$$G(\gamma,n) = \int_{\mathbb{R}} F\Big(\begin{pmatrix}* & *\\ \gamma w_{\mathfrak{a}}^{-\frac12} & \gamma t w_{\mathfrak{a}}^{\frac12}\end{pmatrix}\Big)\mathrm{e}(-nt)\mathrm{d}t.$$

Using the definition (21), we finally obtain

$$U(h,\Psi) = \sum_{n\in\mathbb{Z}}\sum_{\gamma\in\mathcal{C}(\infty,\mathfrak{a})} S_{\infty\mathfrak{a}}(h,n;\gamma)G(\gamma,n). \tag{26}$$

0.1.8. *Localization and preparation of variables.* We recall that $w_{\mathfrak{a}} \asymp_Q qd$. By definition of $F = F_{\Psi,Q}$, we have

$$G(\gamma,n) = \int_{\mathbb{R}} \Psi(tw_{\mathfrak{a}})f\Big(\frac{Q(\gamma,\gamma t w_{\mathfrak{a}})}{qw_{\mathfrak{a}}M}\Big)\mathrm{e}(-nt)\mathrm{d}t.$$

Whenever the integrand is non-zero, we have $|t| \le 2w_{\mathfrak{a}}^{-1}$ and $\gamma \asymp_{Q,f} q(dM)^{\frac12}$. Integrating by parts (*cf.* lemma 5.1 of [Tót00]), we obtain

$$G(\gamma,n) \ll_j (qd)^{j-1}n^{-j} \qquad (j\in\mathbb{N}). \tag{27}$$

The implied constants here also depend on the implied constants in (3) ; this dependency will not be explicited to clarify the notations.

Let $N_1 := qd(Mq)^\eta$. In the RHS of (26), we isolate the contribution of U_0 (resp. U_1) coming from $n=0$ (resp. $|n| > N_1$). The bounds (23), (24) and (27) yields

$$\begin{aligned}|U_0(h,\Psi)| &\ll_{\varepsilon,D} (qd)^{-1}\sum_{\substack{\gamma\in\mathcal{C}(\infty,\mathfrak{a})\\ \gamma\asymp q(dM)^{1/2}}} |S_{\infty\mathfrak{a}}(h,0;\gamma)| \ll (Mqh)^\varepsilon d^{-1}q^{-\frac12}M^{\frac12},\\ |U_1(h,\Psi)| &\ll_{j,D} (qM)^{3/2}\Big(\frac{qd}{N}\Big)^{j-1} \ll_{\eta,D} (qM)^{-10}\end{aligned} \tag{28}$$

choosing $j = j(\eta)$ sufficiently large. Both error terms here are bounded by the RHS of (6).

On another hand, Faà di Bruno's formula shows that the function G satisfies

$$\frac{\partial^{k+\ell}}{\partial x^{\ell_1}\partial y^{\ell_2}}G(x,y)\Big|_{\substack{x=\gamma\\y=n}} \ll_{\ell_1,\ell_2} \gamma^{-\ell_1}(qd)^{-\ell_2-1}.$$

Similarly to (27), this bound also depends on the implicit constants in (3).

We introduce a partition of unity for the variable n,

$$G(\gamma,n)=\sum_{0\leq k\leq K}G_{2^k}(\gamma,n),$$

where $K\leq 2+\log(N_1)/\log 2$, and for $1\leq N\leq N_1$, the function $G_N(\gamma,n)$ is smooth with respect to both variables, vanishes outside $n\in[N/2,2N]$ and satisfies

$$\frac{\partial^{k+\ell}}{\partial x^k\partial y^\ell}G_N(x,y)\Big|_{\substack{x=\gamma\\y=n}} \ll (qd)^{-1}\gamma^{-k}(\min\{qd,N\})^{-\ell} \ll (qM)^{O(\eta\ell)}(qd)^{-1}\gamma^{-k}N^{-\ell}. \tag{29}$$

In accordance with this decomposition, we have

$$\sum_{0<|n|\leq N_1}\sum_{\gamma\in\mathcal{C}(\infty,\mathfrak{a})}S_{\infty\mathfrak{a}}(h,n;\gamma)G(\gamma,n)=\sum_{0\leq k\leq K}V_{2^k}, \tag{30}$$

$$V_N=\sum_{N/2\leq|n|\leq 2N}\sum_{\gamma\in\mathcal{C}(\infty,\mathfrak{a})}S_{\infty\mathfrak{a}}(h,n;\gamma)G_N(\gamma,n). \tag{31}$$

We then let

$$F(x,\xi)=\int_{\mathbb{R}}G_N\Big(\frac{4\pi\sqrt{h|y|}}{x},y\Big)\mathrm{e}(y\xi)\mathrm{d}y,\quad G_N(\gamma,n)=\int_{\mathbb{R}}F\Big(\frac{4\pi\sqrt{h|n|}}{\gamma},\xi\Big)\mathrm{e}(-n\xi)\mathrm{d}\xi.$$

The integral defining F is supported on $y\asymp N$, and when $F(x,\xi)\neq 0$, we necessarily have $x\asymp X:=(hN/q^2dM)^{\frac12}$. Faà di Bruno's formula again implies

$$\partial_{k0}F(x,\xi)\ll_k (qM)^{O(\eta)}X^{-k}\frac{(qd)^{-1}N}{1+(N\xi)^2}.$$

Here the implied constant in $O(\eta)$ is independent of k. We finally let

$$\phi_\xi(x)=X(1+(N\xi)^2)qd(xN)^{-1}(qM)^{-\varpi}F(x,\xi)$$

for some positive number $\varpi=O(\eta)$, so that $x\mapsto\phi_u(x)$ is smooth, compactly supported on $x\asymp X$ and satisfies

$$\sup_{\xi\in\mathbb{R}}\|\phi_\xi^{(j)}\|\ll_j X^{-j}. \tag{32}$$

Here again the implied constants depend on the implied constants in (3). We then have

$$V_N=4\pi(qM)^\varpi d^{-\frac12}M^{\frac12}\int_{\mathbb{R}}\frac{N}{1+(N\xi)^2}W_N(\xi)\mathrm{d}\xi, \tag{33}$$

where we have let

$$W_N(\xi):=\sum_{N/2\leq|n|\leq 2N}a_n\sum_{\gamma\in\mathcal{C}(\infty,\mathfrak{a})}\frac{S^{(\xi)}_{\infty\mathfrak{a}}(h,n;\gamma)}{\gamma}\phi_\xi\Big(\frac{4\pi\sqrt{h|n|}}{\gamma}\Big), \tag{34}$$

as well as $a_n=\sqrt{|n|/N}$, and where we have incorporated the factor $\mathrm{e}(-n\xi)$ in the scaling matrix of ∞ (which is indicated by the notation $S^{(\xi)}_{\infty\mathfrak{a}}$).

0.1.9. *Using Kuznetsov's formula.* We bound W_N separately for each ξ. We omit the quantity ξ from the notation, and we will use from (a_n) and ϕ only the bounds $|a_n| \leq 1$, the bounds (32) and the fact that $\phi(x) \neq 0$ implies $x \asymp X$, where we recall that $X \asymp (hN/q^2dM)^{\frac{1}{2}}$.

For each $n \in [N/2, 2N]$, we apply Kuznetsov's formula (lemma 4.5 of [Dra17], with $\kappa = 0$). We obtain

$$\sum_{\gamma \in \mathcal{C}(\infty,\mathfrak{a})} \frac{S_{\infty\mathfrak{a}}(h,n;\gamma)}{\gamma}\phi\Big(\frac{4\pi\sqrt{hn}}{\gamma}\Big) = \mathscr{H}^+_{h,n} + \mathscr{E}^+_{h,n} + \mathscr{M}^+_{h,n}, \qquad (n > 0)$$

$$\sum_{\gamma \in \mathcal{C}(\infty,\mathfrak{a})} \frac{S_{\infty\mathfrak{a}}(h,n;\gamma)}{\gamma}\phi\Big(\frac{4\pi\sqrt{h|n|}}{\gamma}\Big) = \mathscr{E}^-_{h,n} + \mathscr{M}^-_{h,n}, \qquad (n < 0)$$

where

$$\mathscr{M}^+_{h,n} = \sum_{f \in \mathscr{B}(q,\chi)} \frac{\widetilde{\phi}(t_f)}{\cosh(\pi t_f)}(hn)^{\frac{1}{2}}\overline{\rho_{f\infty}(h)}\rho_{f\mathfrak{a}}(n),$$

and $\mathscr{M}^-_{h,n}$, $\mathscr{E}^\pm_{h,n}$, $\mathscr{H}^+_{h,n}$ are given by similar expression. Here, the set $\mathscr{B}(q,\chi)$ denotes an orthonormal basis of Maass cusp forms f, each being an eigenfunction of the hyperbolic Laplacian, with associated eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$ and Fourier coefficients $\rho_{f\mathfrak{a}}(n)$. We refer to section 4.1.2 of [Dra17] for the associated definitions and normalisation. We have $t_f \in \mathbb{R} \cup [-i\theta, i\theta]$, where we recall that $\theta \leq 7/64$ by Kim and Sarnak [Kim03], and that the Selberg-Ramanujan conjecture predicts that $\theta = 0$. The transform $\widetilde{\phi}$ in the expression above is given by

$$\widetilde{\phi}(t) = \frac{2\pi i}{\sinh(\pi t)}\int_0^\infty (J_{2it}(x) - J_{-2it}(x))\phi(x)\frac{\mathrm{d}x}{x}$$

where $J_\nu(x)$ is the J-Bessel function. The transform $\widetilde{\phi}$ satisfies the bounds in lemma 4.4 of [Dra17] (see lemma 2.4 of [Top15] for stronger bounds). In the present case, we have $X \ll (qM)^{\eta/2}$, so that

$$|\widetilde{\phi}(t)| \ll \begin{cases} (qM)^{2\eta}(1+|t|)^{-3}, & t \in \mathbb{R}, \\ (Mq)^{\eta/2}(q^2dM/hN)^{|t|}, & t \in [-i/4, i/4]. \end{cases}$$

The quantity $\mathscr{E}^\pm_{h,n}$ (resp. $\mathscr{H}^+_{h,n}$) corresponds to the contribution of non-holomorphic Eisenstein series (resp. to the contribution of holomorphic cusp forms of weight ≥ 2). We study in detail $\mathscr{M}^+$, the other terms being analyzed in a similar manner.

Our treatment differs depending on whether we average over h or not.

0.1.10. *The case (h,q) fixed.* We separate in $\mathscr{M}^+_{h,n}$ the contribution of functions $f \in \mathscr{B}(q,\chi)$ with $t_f \in \mathbb{R}$, from those with $t_f \in i\mathbb{R}$. According to this decomposition we write

$$W_N = \sum_{N/2 \leq n \leq 2N} a_n \mathscr{M}^+_{h,n} = \mathscr{M}^{\mathrm{reg}}_{h,N} + \mathscr{M}^{\mathrm{exc}}_{h,N}, \tag{35}$$

where the notation corresponds to "regular" and "exceptional". The Cauchy-Schwarz inequality yields

$$|\mathscr{M}^{\mathrm{reg}}_{h,N}| \leq (\mathscr{M}^{\mathrm{reg}}_h \mathscr{M}^{\mathrm{reg}}_N)^{\frac{1}{2}}, \tag{36}$$

with

$$\mathscr{M}_h^{\mathrm{reg}} := \sum_{\substack{f\in\mathscr{B}(q,\chi)\\ t_f\in\mathbb{R}}} \frac{|\widetilde{\phi}(t_f)|}{\cosh(\pi t_f)} h|\rho_{f\infty}(h)|^2,$$

$$\mathscr{M}_N^{\mathrm{reg}} := \sum_{\substack{f\in\mathscr{B}(q,\chi)\\ t_f\in\mathbb{R}}} \frac{|\widetilde{\phi}(t_f)|}{\cosh(\pi t_f)} \Big| \sum_{N/2\le n\le 2N} a_n n^{\frac12}\rho_{f\mathfrak{a}}(n)\Big|^2.$$

To bound $\mathscr{M}_h^{\mathrm{reg}}$, we use lemma 2.7 of [Top15], so that

$$\mathscr{M}_h^{\mathrm{reg}} \ll_\varepsilon (qhM)^\varepsilon \Big\{ 1 + (qd,h)^{\frac12} \frac{h^{\frac12}}{qd^{\frac12}} \Big\}.$$

To bound $\mathscr{M}_N^{\mathrm{reg}}$, we use the large sieve inequality (in our case, proposition 4.7 of [Dra17])

$$\mathscr{M}_N^{\mathrm{reg}} \ll_\varepsilon (qM)^\varepsilon N \Big\{ 1 + \frac{N}{qd^{\frac12}} \Big\}. \tag{37}$$

Our hypotheses $h \ll q$ and $N \le N_1$ then imply

$$\mathscr{M}_{h,N}^{\mathrm{reg}} \ll_\eta (qM)^{O(\eta)} q^{\frac12} d^{\frac34}. \tag{38}$$

For each h, we have by the Cauchy-Schwarz inequality

$$|\mathscr{M}_{h,N}^{\mathrm{exc}}| \ll (\mathscr{M}_h^{\mathrm{exc}} \mathscr{M}_N^{\mathrm{exc}})^{\frac12}, \tag{39}$$

$$\mathscr{M}_h^{\mathrm{exc}} := \sum_{\substack{f\in\mathscr{B}(q,\chi)\\ t_f\in i\mathbb{R}}} |\widetilde{\phi}(t_f)|^2 h |\rho_{f\infty}(h)|^2, \quad \mathscr{M}_N^{\mathrm{exc}} := \sum_{\substack{f\in\mathscr{B}(q,\chi)\\ t_f\in i\mathbb{R}}} \Big| \sum_{N/2\le n\le 2N} a_n n^{\frac12}\rho_{f\mathfrak{a}}(n)\Big|^2. \tag{40}$$

The large sieve again yields

$$\mathscr{M}_N^{\mathrm{exc}} \ll_\varepsilon (qM)^\varepsilon N \Big\{ 1 + \frac{N}{qd^{\frac12}} \Big\}. \tag{41}$$

For $\mathscr{M}_h^{\mathrm{exc}}$, we use lemma 2.9 of [Top15],

$$\mathscr{M}_h^{\mathrm{exc}} \ll_\eta (qhM)^{O(\eta)} \{(qd,h)MN^{-1}\}^{2\theta}.$$

Our hypothesis $N \le N_1$ then implies

$$\mathscr{M}_{h,N}^{\mathrm{exc}} \ll_\eta (qhM)^{O(\eta)} d^{\frac14} (qd,h)^\theta M^\theta (qd)^{\frac12-\theta}.$$

The right-hand side here is larger than that obtain in (38). Therefore, we have

$$\sum_{N/2\le n\le 2N} a_n \mathscr{M}_{h,n}^{+} \ll_\eta (qhM)^{O(\eta)} d^{\frac14} (qd,h)^\theta M^\theta (qd)^{\frac12-\theta}.$$

The same bounds holds for $\mathscr{M}_{h,n}^{-}$, whereas the other terms $\mathscr{E}^{\pm}$ and $\mathscr{H}^{+}$, are of the order of the RHS of (38). We therefore have

$$W_N \ll_\eta (qM)^{O(\eta)} d^{\frac14} (qd,h)^\theta M^\theta (qd)^{\frac12-\theta}.$$

We insert this in (33) then (30), which yields, with the bounds (28) and choosing $\eta > 0$ arbitrarily small,

$$U(h,\Psi) \ll_\varepsilon (qM)^\varepsilon d^{-\frac14} (qd,h)^\theta M^\theta (qd)^{\frac12-\theta}.$$

We insert this again in (18), using point (i) of Lemma 6 to bound the sum over σ. We get

$$T_Q(h,\Psi) \ll_\varepsilon (qM)^\varepsilon d^{\frac34} (qd,h)^\theta M^\theta (qd)^{\frac12-\theta},$$

which gives the bound (6) by using (17), (12) and (9) successively.

0.1.11. *Bound on average over h.* In this section we justify the bound (7). When $H \leq qd^{\frac{1}{2}}$, we may simply take the average over h of the bound (6) established in the previous sections. We henceforth assume that $H > qd^{\frac{1}{2}}$.

Let $(c_h) \in \mathbb{C}^{\mathbb{N}}$, $|c_h| \leq 1$ be a sequence with

$$\Big| \sum_{(m,\Omega)\in\mathcal{D}} f\Big(\frac{m}{M}\Big)\mathrm{e}\Big(\frac{h\Omega}{mq}\Big)\Big| = c_h \sum_{(m,\Omega)\in\mathcal{D}} f\Big(\frac{m}{M}\Big)\mathrm{e}\Big(\frac{h\Omega}{mq}\Big).$$

The coefficients (c_h) depend at most on $(h, q, d, \lambda, \omega, M, f)$.

Recalling the definition (35), it will suffice to prove that

$$\mathscr{M}^{\mathrm{reg}}_{H,N} := \frac{1}{H} \sum_{H<h\leq 2H} c_h \mathscr{M}^{\mathrm{reg}}_{h,N} \ll_\eta (qHM)^{O(\eta)} d^{\frac{3}{4}} q^{\frac{1}{2}}, \tag{42}$$

$$\mathscr{M}^{\mathrm{exc}}_{H,N} := \frac{1}{H} \sum_{H<h\leq 2H} c_h \mathscr{M}^{\mathrm{exc}}_{h,N} \ll_\eta (qHM)^{O(\eta)} d^{\frac{1}{4}} M^\theta (qd)^{\frac{1}{2}-\theta}. \tag{43}$$

The bounds (37) and (41) are valid on average over h, since they do not depend on h.

In the case of (42), a reasonning similar to (36) reduces the problem to the estimation of

$$\mathscr{M}^{\mathrm{reg}}_{H} := H^{-2} \sum_{\substack{f\in\mathscr{B}(q,\chi)\\ t_f\in\mathbb{R}}} \frac{|\widetilde{\phi}(t_f)|}{\cosh(\pi t_f)} \Big| \sum_{H<h\leq 2H} c_h \sqrt{h} \rho_{f\infty}(h)\Big|^2.$$

The large sieve inequality yields

$$\mathscr{M}^{\mathrm{reg}}_{H} \ll_\varepsilon (qHM)^\varepsilon H^{-1}\Big\{1 + \frac{H}{qd^{\frac{1}{2}}}\Big\} \ll_\varepsilon (qHM)^\varepsilon,$$

whence we deduce the bound (42).

As concerns (43), by reasonning similarly to (41), the problem is reduced to considering

$$\mathscr{M}^{\mathrm{exc}}_{H} := H^{-2} \sum_{\substack{f\in\mathscr{B}(q,\chi)\\ t_f\in i\mathbb{R}}} |\widetilde{\phi}(t_f)|^2 \Big| \sum_{H<h\leq 2H} c_h \sqrt{h} \rho_{f\infty}(h)\Big|^2.$$

We use the large sieve inequality for the exceptional spectrum, in our case lemma 4.8 of [Dra17], using the bound of Kim-Sarnak (see the remark preceding section 4.3 of [Dra17]). We obtain

$$\begin{aligned}\mathscr{M}^{\mathrm{exc}}_{H} &\ll_\eta H^{-1}(qM)^{O(\eta)}\Big(1 + \Big(\frac{qM}{N}\Big)^{2\theta}\Big)\Big(1 + d^{\frac{1}{2}}\Big(\frac{H}{qd}\Big)^{1-2\theta}\Big)\\ &\ll_\eta (qM)^{O(\eta)}(q^2 d)^{2\theta-\frac{1}{2}}\Big(\frac{M}{N}\Big)^{2\theta}.\end{aligned}$$

This plainly suffices to prove (43). Formula (7) is deduced in a way similar to the case of fixed h.

0.1.12. *Bound on average over h and q.* Suppose now that $d=1$, and $c_h = \mathrm{e}(th)\mathbf{1}_{h\in\mathcal{I}}$ for some interval $\mathcal{I}\subset[H,2H]$. We follow the arguments from the previous sections, encoding the factor $\mathrm{e}(th)$ in the scaling matrix of ∞, which brings us to the estimation of

$$\mathscr{M}_H^{\mathrm{exc}}(q) = H^{-2}\sum_{\substack{f\in\mathscr{B}(q,\mathbf{1})\\ t_f\in i\mathbb{R}}} |\widetilde{\phi}(t_f)|^2\Big|\sum_{h\in\mathcal{I}}\sqrt{h}\rho_{f\infty}(h)\Big|^2.$$

We sum this over $q\in[Q,2Q]$, and use the weighted large sieve inequality of Deshouillers-Iwaniec, Theorem 7 of [DI83]. We obtain

$$\frac{1}{Q}\sum_{Q<q\leq 2Q}\mathscr{M}_H^{\mathrm{exc}}(q) \ll_{\varepsilon,\eta} M^{O(\eta)}H^{-1+\varepsilon}\Big\{1+\frac{H}{Q}+\Big(\frac{M}{N}\Big)^{2\theta}\Big\}.$$

With 2θ remplaced with $\frac12$, this follows directly from Theorem 7 of [DI83]. The previous bound is easily justified by noting that at the conclusion of the proof of Theorem 7, page 278 of [DI83], the quantity $\sqrt{Y/Y_1}$ may be replaced by $(Y/Y_1)^{2\theta}$. The conclusion of the proof follows in a way identical to the case of fixed h.

Remarks.

- When H is large, the error term we obtain is slightly better than that announced in (7). This has no bearing on the application we consider here.
- The factors h and H in the first terms of the RHS of (6)-(8) may be improved by using integration by parts instead of the trivial approximation (16).

0.2. **Proof of Lemma 4.** From Lemma 5, we deduce by a standard Fourier analytic technique the estimate

$$\begin{aligned}&P_f(M;q,r,d,\lambda,\omega,\alpha,\beta)\\ &\qquad=(\beta-\alpha)P_f(M;q,r,d,\lambda,\omega,0,1)+O_{\varepsilon,D,f}((qM)^{\varepsilon}d^{\frac34}(qd)^{\frac12-\theta}M^{\frac12+\theta}).\end{aligned}\tag{44}$$

We omit the details, which are similar to pages 179 and 180 of [Iwa78]. The only difference with our treatment lies in the additional terms h and H in the RHS of (6) and (7), which forces the choice $\Delta=(q+M^{\frac12})^{-1}$ in the argument of Iwaniec. This induces an additional error term

$$\ll \Delta^{-1}+M\Delta\ll q+M^{\frac12}\ll q^{\frac12-\theta}M^{\frac12+\theta},$$

which is acceptable.

We therefore focus on the treatment of the main term. We require the following lemma.

Lemma 7. *Let $x\in\mathbb{R}$ with $x\geq 1$, $D\in\mathbb{Z}$ which is not a perfect square, $(q,d)\in\mathbb{N}^2$ with $q\geq 1$, $(q,2D)=1$, $d|q$ and $\lambda \pmod d$ with $(\lambda,d)=1$. Let $\chi_D=(\frac{D}{\cdot})$ be the Kronecker symbol, and $\varkappa_D(n):=(1*\chi_D)(n)$. Then*

$$\sum_{\substack{n\leq x\\ (n,q)=1\\ n\equiv\lambda \pmod d}}\varkappa_D(n)=\frac{x}{\varphi(d)}\frac{\varphi(q)}{q}\prod_{p|q}\Big(1-\frac{\chi_D(p)}{p}\Big)L(1,\chi_D)+O_{\varepsilon,D}(x^{\frac12}q^{\varepsilon}).$$

Proof. This follows easily from the Dirichlet hyperbola method. ☐

Recall that for $p \nmid 2D$, we have $\rho(p) = 1 + (\frac{D}{p}) = \varkappa_D(p)$. We write $\rho = \varkappa_D * h_D$, in such a way that the function h_D satisfies $\sum_\ell |h_D(\ell)|\ell^{-\frac12-\varepsilon} \ll_{\varepsilon,D} 1$. When $M \geq 1$ and $(\lambda, d) = 1$, using Lemma 7 and integration by parts, we deduce

$$\sum_{\substack{(m,q)=1\\ m\equiv\lambda \pmod d}} f\Big(\frac{m}{M}\Big)\rho(m) = \sum_{\substack{(\ell,q)=1\\ \ell\ll M}} h_D(\ell) \sum_{\substack{(n,q)=1\\ n\equiv\lambda\bar\ell \pmod d}} f\Big(\frac{n\ell}{M}\Big)\varkappa_D(n)$$

$$= \frac{1}{\varphi(d)}\frac{\varphi(q)}{q}L(1,\chi_D)M\hat f(0)\sum_{(\ell,q)=1}\frac{h_D(\ell)}{\ell} + O_{\varepsilon,D,f}(q^\varepsilon M^{\frac12+\varepsilon}). \tag{45}$$

Let

$$C_D := L(1,\chi_D)\sum_{\ell\geq1}\frac{h_D(\ell)}{\ell} = \sum_{\ell\geq1}\frac{(\rho*\mu)(\ell)}{\ell}.$$

We obtain

$$L(1,\chi_D)\frac{\varphi(q)}{q}\sum_{(\ell,q)=1}\frac{h_D(\ell)}{\ell} = C_D\prod_{p|q}\Big(1+\frac1p\Big)^{-1}. \tag{46}$$

We return now to the estimation of the main term in the RHS of (44). The Chinese remainder theorem and the relations (45) and (46) with q replaced by qr yield

$$\begin{aligned}P_f(M;q,r,d,\lambda,\omega,0,1) &= \sum_{\substack{(m,qr)=1\\ m\equiv\lambda \pmod d}} f\Big(\frac{m}{M}\Big)\sum_{\substack{\Omega \pmod{qm}\\ \Omega^2\equiv D \pmod{qm}\\ \Omega\equiv\omega \pmod d}} 1\\ &= \rho_{\omega,d}(q)\sum_{\substack{(m,qr)=1\\ m\equiv\lambda \pmod d}} f\Big(\frac{m}{M}\Big)\rho(m)\\ &= C_D\prod_{p|qr}\Big(1+\frac1p\Big)^{-1}\frac{\rho_{\omega,d}(q)}{\varphi(d)}M\hat f(0) + O_{\varepsilon,D,f}(M^{\frac12+\varepsilon}q^\varepsilon),\end{aligned}$$

where we have let, for all $\omega \pmod d$ with $\omega^2 \equiv D \pmod d$,

$$\rho_{\omega,d}(q) = \sum_{\substack{\Omega \pmod q\\ \Omega^2\equiv D \pmod q\\ \Omega\equiv\omega \pmod d}} 1.$$

It is easy to see that $\rho_{\omega,d}(q) = \rho(q)$ if $d = 1$, and for all $p \nmid 2D$, $1 \leq \delta \leq \nu$, $\rho_{\omega,p^\delta}(p^\nu) = 1$ by Hensel's lemma. We deduce that $\rho_{\omega,d}(q) = \rho(q/(q,d^\infty))$ independently of ω. This concludes the proof of Lemma 4.

0.3. **Proof of Proposition 3.**

0.3.1. *First reduction.* We remark first that the trivial bound $x^{2+\varepsilon}/M$ for the LHS of (2) allows us to assume without loss that $x \geq M$.

To simplify the proof of Proposition 3, we first justify that we may assume the sequence (b_n) to be supported on odd integers coprime to D. Suppose first, then, that the estimate (2) holds for such sequences. Letting

$$r_D(x;q) := \sum_{\substack{k\in\mathbb{N}\\ q|k^2-D}} V\Big(\frac{k}{x}\Big) - x\hat V(0)\frac{\rho(q)}{q}, \tag{47}$$

we have

$$\sum_{M<m\le 2M}\Big|\sum_{\substack{N<n\le 2N\\(n,m)=1}} b_n r_D(x;mn)\Big| = \sum_{M<m\le 2M}\Big|\sum_{\substack{1\le v\le 2N\\ v|(2D)^\infty}}\sum_{\substack{N/v<n\le 2N/v\\(n,2Dm)=1}} b_{vn} r_D(x;vmn)\Big|$$
$$\le \sum_{\substack{v\le 2N\\ v|(2D)^\infty}}\sum_{vM<m\le 2vM}\Big|\sum_{\substack{N/v<n\le 2N/v\\(n,2Dm)=1}} b_{vn} r_D(x;mn)\Big|.$$

The bound (2) applied for each c in the RHS yields the desired bound.

We therefore assume in what follows that (b_n) is supported on integers n such that $(n,2D)=1$.

0.3.2. *Interpreting a congruence condition.* We follow the arguments in pages 180-183 of [Iwa78]. To do this, we modify the construction of the class $c \pmod{[n_1,n_2]}$, page 183 of [Iwa78], to deal with the fact that in our case, the sequence (b_n) is not assumed to be supported on squarefree integers.

Lemma 8. *Let $m,n_1,n_2,\ell_1,\ell_2\ge 1$ be given, with $(2mD,n_1n_2)=1$. Let*

$$d=(n_1,n_2)/(n_1,n_2,\ell_1-\ell_2),$$

and suppose that

$$(m(\ell_1-\ell_2))^2\equiv 4D \pmod d. \tag{48}$$

Then there exists $c\in\mathbb{Z}$, with $0\le c<[n_1,n_2]$, such that the sets

$$\mathcal{D}_1=\Big\{v\in\mathbb{Z}\cap[0,m):\ \begin{matrix} v^2\equiv D \pmod m\\ (m\ell_j+v)^2\equiv D \pmod{n_j}\quad (j\in\{1,2\})\end{matrix}\Big\}$$

and

$$\mathcal{D}_2=\Big\{\Omega\in\mathbb{Z}\cap[cm,(c+1)m):\ \begin{matrix}\Omega^2\equiv D \pmod{m[n_1,n_2]}\\ \Omega\equiv m(c-\tfrac12(\ell_1+\ell_2)) \pmod d\end{matrix}\Big\}$$

are in bijection.

Remark. The sets $\mathcal{D}_1$ and $\mathcal{D}_2$ are empty if the condition (48) is not satisfied.

Proof. Let

$$n_j=\prod_p p^{\nu_j(p)}\qquad (j\in\{1,2\}),$$

We define $c\in\mathbb{Z}$, $0\le c<[n_1,n_2]$ as the unique integers satisfying, for all p,

$$c\equiv\begin{cases}\ell_1 \pmod{p^{\nu_1(p)}}, & \text{si } \nu_1(p)\ge\nu_2(p),\\ \ell_2 \pmod{p^{\nu_2(p)}} & \text{sinon.}\end{cases}$$

To each $v\in\mathbb{Z}\cap[0,m)$, we associate $\Omega(v)=cm+v\in[cm,(m+1)c)$. This map is bijective, and it will suffice to show that $\Omega(\mathcal{D}_1)=\mathcal{D}_2$. Suppose $v\in\mathcal{D}_1$, and let $\Omega=\Omega(v)$. Since $(m,[n_1,n_2])=1$, it suffices to prove the congruence $\Omega^2\equiv D$ modulo m and $[n_1,n_2]$, separately. We have $\Omega\equiv v \pmod m$, which yields $\Omega^2\equiv D \pmod m$. For all p, we have

$$\Omega\equiv \ell_j m+v \pmod{p^{\nu_j(p)}},$$

with $j=1$ if $\nu_1(p)\ge\nu_2(p)$, and $j=2$ otherwise. In both cases, we obtain $\Omega^2\equiv D \pmod{p^{\nu_j(p)}}$, therefore $\Omega^2\equiv D \pmod{[n_1,n_2]}$. The condition $\Omega\equiv m(c-\frac12(\ell_1+\ell_2)) \pmod d$ easily follows from the fact that

$$(m\ell_1+v)^2\equiv(m\ell_2+v)^2 \pmod{(n_1,n_2)}.$$

Suppose conversely that $\Omega \in \mathcal{D}_2$ is given, and let $v = \Omega - mc$. The congruence $v^2 \equiv D \pmod m$ is then immediate. Next, let p be fixed, let $\nu_j = \nu_j(p)$ and suppose $\nu_1 \geq \nu_2$ (the complementary case is treated in an identical way). We therefore have

$$c \equiv \ell_1 \pmod{p^{\nu_1}}, \qquad \Omega^2 \equiv D \pmod{p^{\nu_1}},$$

which yields directly the congruence $(m\ell_1 + v)^2 \equiv D \pmod{p^{\nu_1}}$. On another hand, we have

$$(m\ell_2 + v)^2 \equiv \Omega^2 - 2m(\ell_1 - \ell_2)\Omega + (m(\ell_1 - \ell_2))^2 \pmod{p^{\nu_2}}.$$

By hypothesis, we have $\Omega^2 \equiv D \pmod{p^{\nu_2}}$. Then,

$$\Omega \equiv m(c - \tfrac{1}{2}(\ell_1 + \ell_2)) \equiv \tfrac{1}{2}m(\ell_1 - \ell_2) \pmod{\tfrac{p^{\nu_2}}{(p^{\nu_2}, \ell_1 - \ell_2)}},$$

which yields

$$2m(\ell_1 - \ell_2)\Omega \equiv (m(\ell_1 - \ell_2))^2 \pmod{p^{\nu_2}}.$$

We deduce $(m\ell_2 + v)^2 \equiv D \pmod{p^{\nu_2}}$. We have therefore obtained $v \in \mathcal{D}_1$. □

0.3.3. *Using the dispersion method.* We expand the square in the LHS of (2). In agreement with [Iwa78], we let

$$Y(m) := \sum_{\substack{N < n \leq 2N \\ (n,m)=1}} b_n \frac{\rho(n)}{n}.$$

Let also the smooth function $f : \mathbb{R} \to \mathbb{R}$ be given and such that $\mathbf{1}_{1 \leq t \leq 2} \leq f(t) \leq \mathbf{1}_{1/2 \leq t \leq 3}$. Finally, we recall the notation (47). The LHS of (2) is bounded above by

$$\sum_m f\Big(\frac{m}{M}\Big) \Big| \sum_{\substack{N < n \leq 2N \\ (n,m)=1}} b_n r_D(x; mn) \Big|^2 = S_1 - 2x\overline{\widehat{V}(0)} \operatorname{Re} S_2 + |x\widehat{V}(0)|^2 S_3, \tag{49}$$

with

$$S_j = \sum_m f\Big(\frac{m}{M}\Big) \sum_{\substack{0 \leq v < m \\ v^2 \equiv D \pmod m}} T_j(m),$$

and

$$T_1(m) = \sum\sum_{\substack{N < n_1, n_2 \leq 2N \\ (n_1 n_2, m)=1}} b_{n_1} \overline{b_{n_2}} \sum\sum_{\substack{k_1, k_2 \in \mathbb{N} \\ k_j \equiv v \pmod m \\ k_j^2 \equiv D \pmod{n_j}}} V\Big(\frac{k_1}{x}\Big) \overline{V\Big(\frac{k_2}{x}\Big)},$$

$$T_2(m) = \frac{\overline{Y(m)}}{m} \sum_{\substack{N < n \leq 2N \\ (n,m)=1}} b_n \sum_{\substack{k \in \mathbb{N} \\ k \equiv v \pmod m \\ k^2 \equiv D \pmod n}} V\Big(\frac{k}{x}\Big), \qquad T_3(m) = \Big(\frac{Y(m)}{m}\Big)^2.$$

0.3.4. *Estimation of S_3.* We have

$$S_3 = \frac{1}{M^2} \sum\sum_{N < n_1, n_2 \leq 2N} b_{n_1} \overline{b_{n_2}} \frac{\rho(n_1)\rho(n_2)}{n_1 n_2} \sum_{(m, n_1 n_2)=1} \frac{M^2}{m^2} f\Big(\frac{m}{M}\Big) \rho(m).$$

With $g_1(t) = t^{-2} f(t)$, the m-sum in the RHS equals

$$P_{g_1}(M; 1, n_1 n_2, 1, 1, 1, 0, 1).$$

We therefore obtain

$$S_3 = P_3 + O_{\varepsilon,D}(x^{\varepsilon} M^{-\frac{3}{2}+\theta}), \tag{50}$$

with

$$P_3 = C_D M^{-1}\Big(\int_{\mathbb{R}} t^{-2} f(t)\mathrm{d}t\Big) \sum\sum_{N<n_1,n_2\le 2N} b_{n_1}\overline{b_{n_2}} A(n_1n_2)\frac{\rho(n_1)\rho(n_2)}{n_1n_2}.$$

0.3.5. *Estimation of S_2.* We have

$$S_2 = \sum\sum_{N<n_1,n_2\le 2N} b_{n_1}\overline{b_{n_2}}\frac{\rho(n_2)}{n_2} \sum_{(m,n_1n_2)=1} \frac{1}{m} f\Big(\frac{m}{M}\Big) \sum_{\substack{0\le v<m \\ v^2\equiv D \pmod m}} \sum_{\substack{k\in\mathbb{N} \\ k\equiv v \pmod m \\ k^2\equiv D \pmod{n_1}}} V\Big(\frac{k}{x}\Big).$$

We write $k = m\ell + v$ with $\ell \ge 0$ and $\ell \ll x/m$. We therefore have

$$V\Big(\frac{m\ell+v}{x}\Big) = V\Big(\frac{m\ell}{x}\Big) + O\Big(\frac{m}{x}\Big),$$

which yields, similarly to [Iwa78, formula (11)], the approximation $S_2 = S_2' + O(x^\varepsilon)$ with

$$S_2' = \sum\sum_{N<n_1,n_2\le 2N} b_{n_1}\overline{b_{n_2}}\frac{\rho(n_2)}{n_2} \sum_{(m,n_1n_2)=1} \frac{1}{m} f\Big(\frac{m}{M}\Big) \sum\sum_{\substack{\ell\ge 0 \\ 0\le v<m \\ v^2\equiv D \pmod m \\ (m\ell+v)^2\equiv D \pmod{n_1}}} V\Big(\frac{m\ell}{x}\Big).$$

The condition on the supports of f and V imply that the integers ℓ giving a non-trivial contribution to S_2' come from an interval of integers I such that $\ell \asymp x/M$ for each $\ell \in I$. For all n_2 with $\rho(n_2) \ne 0$, we let $n_2' = n_2/(n_2, n_1^\infty)$. Let $c \in \mathbb{N} \cap [0, n_1)$ be the unique integer satisfying $c \equiv \ell \pmod{n_1}$. We have a bijection

$$\Big\{v \in \mathbb{N}\cap[0,m) : \begin{array}{l} v^2 \equiv D \pmod m \\ (m\ell+v)^2 \equiv D \pmod{n_1} \end{array}\Big\}$$
$$\longrightarrow \Big\{\Omega \in \mathbb{N}\cap[0,mn_1) : \begin{array}{l} \Omega^2 \equiv D \pmod{mn_1} \\ cm \le \Omega < (c+1)m \end{array}\Big\}$$

given by $v \mapsto mc + v$. Therefore,

$$S_2' = \frac{1}{M}\sum\sum_{N<n_1,n_2\le 2N} b_{n_1}\overline{b_{n_2}}\frac{\rho(n_2)}{n_2\rho(n_2/(n_2,n_1^\infty))} \sum_{\ell\in I} \sum_{(m,n_1n_2)=1} g_{2,\ell}\Big(\frac{m}{M}\Big) \sum_{\substack{\Omega\in\mathbb{N} \\ \Omega^2\equiv D \pmod{mn_1} \\ cm\le\Omega<(c+1)m}} 1.$$

where $g_{2,\ell}(t) = t^{-1}f(t)V(t\ell M/x)$, which satisfies the hypothesis (3). The sum over (m,Ω) is exactly $P_{g_{2,\ell}}(M; n_1, n_2', 1, 1, 1, \frac{c}{n_1}, \frac{c+1}{n_1})$, Lemma 4 therefore yields

$$S_2' = P_2 + O_{\varepsilon,D}(x^\varepsilon N^{-\frac{3}{2}-\theta} M^{-\frac{1}{2}+\theta}),$$

with

$$P_2 = C_D \sum\sum_{N<n_1,n_2\le 2N} b_{n_1}\overline{b_{n_2}}\frac{\rho(n_1)\rho(n_2)}{n_1n_2} A(n_1n_2) \int t^{-1} f(t) \sum_{\ell\in\mathbb{Z}} V\Big(\frac{\ell t M}{x}\Big)\mathrm{d}t.$$

Uniformly for $t \in \operatorname{supp} f$, we use

$$\sum_{\ell\in\mathbb{Z}} V\Big(\frac{\ell t M}{x}\Big) = \frac{x}{Mt}\widehat{V}(0) + O(1),$$

which yields $P_2 = x\widehat{V}(0)P_3 + O_{\varepsilon,D}(x^\varepsilon)$, and finally

$$S_2 = x\widehat{V}(0)P_3 + O_{\varepsilon,D}(x^\varepsilon\{1 + N^{-\frac{3}{2}-\theta}M^{-\frac{1}{2}+\theta}\}). \tag{51}$$

0.3.6. *Estimation of S_1 and conclusion.* In the sum S_1, we let $k_j = m\ell_j + v$ be given with $\ell_j \geq 0$, so that

$$S_1 = \sum\sum_{N<n_1,n_2\leq 2N} b_{n_1}\overline{b_{n_2}} \sum\sum_{\ell_1,\ell_2\geq 0} \sum_{(m,n_1n_2)=1} f\Big(\frac{m}{M}\Big) \sum_{\substack{0\leq v<m \\ v^2\equiv D \pmod m \\ (m\ell_j+v)^2\equiv D \pmod{n_j}}} V\Big(\frac{m\ell_1+v}{x}\Big)\overline{V\Big(\frac{m\ell_2+v}{x}\Big)}.$$

We replace the product $V(\dots)\overline{V(\dots)}$ by $V(m\ell_1/x)\overline{V(m\ell_2/x)}$. The error induced in S_1 by this replacement is $O_{\varepsilon,D}(x^{1+\varepsilon})$, so that $S_1 = S_1' + O_{\varepsilon,D}(x^{1+\varepsilon})$, with

$$\begin{aligned} S_1' = \sum\sum_{N<n_1,n_2\leq 2N} b_{n_1}\overline{b_{n_2}} \sum\sum_{\ell_1,\ell_2\geq 0} \sum_{(m,n_1n_2)=1} f\Big(\frac{m}{M}\Big)\times \\ \times V\Big(\frac{m\ell_1}{x}\Big)\overline{V\Big(\frac{m\ell_2}{x}\Big)} \sum_{\substack{0\leq v<m \\ v^2\equiv D \pmod m \\ (m\ell_j+v)^2\equiv D \pmod{n_j}}} 1 \end{aligned} \tag{52}$$

For each (n_1,n_2,ℓ_1,ℓ_2), the sum over v is expressed by means of Lemma 8. We let $q = [n_1,n_2]$, $d = (n_1,n_2)/(n_1,n_2,\ell_1-\ell_2)$, and

$$\mathcal{L} = \{\lambda \pmod d : \ (\lambda(\ell_1-\ell_2))^2 \equiv 4D \pmod d\}.$$

Since $(d,2D)=1$, we have $\mathcal{L}=\varnothing$ si $(\ell_1-\ell_2,d)>1$, and $|\mathcal{L}| = \rho(d)$ otherwise. The sum over (ℓ_1,ℓ_2) is therefore restricted to $(\ell_1-\ell_2,d)=1$. The sum over (m,v) in the RHS of (52) equals

$$\sum_{\lambda\in\mathcal{L}} P_{g_3}\Big(M;q,d,\lambda,\omega_\lambda,\frac{c}{q},\frac{c+1}{q}\Big) \qquad (\omega_\lambda = \lambda(c-\tfrac12(\ell_1+\ell_2))),$$

with $g_3(t) = f(t)V(t\ell_1M/x)\overline{V(t\ell_2M/x)}$. Since $|\mathcal{L}| = \rho(d)$ and $\rho(d)\rho(q/(q,d^\infty)) = \rho(q)$, Lemma 4 yields

$$S_1' = P_1 + O_{\varepsilon,D}\Big(x^{1+\varepsilon} + x^{2+\varepsilon}\Big(\frac{N^2}{M}\Big)^{\frac32-\theta}\Big),$$

with

$$P_1 = C_D M \sum\sum_{N<n_1,n_2\leq 2N} b_{n_1}\overline{b_{n_2}} \sum\sum_{\substack{\ell_1,\ell_2\geq 0 \\ (\ell_1-\ell_2,d)=1}} \frac{\rho(q)}{q}\frac{A(q)}{\varphi(d)}\int_{\mathbb{R}} f(t)V\Big(\frac{t\ell_1M}{x}\Big)\overline{V\Big(\frac{t\ell_2M}{x}\Big)}\mathrm{d}t. \tag{53}$$

We denote temporarily $n_0 = (n_1,n_2)$. Recall that $d = n_0/(n_0,\ell_1-\ell_2)$. For $X \gg 1$, we have

$$\begin{aligned} \sum\sum_{\substack{\ell_1,\ell_2\in\mathbb{N} \\ (\ell_1-\ell_2,n_0)=n_0/d \\ (\ell_1-\ell_2,d)=1}} V\Big(\frac{\ell_1}{X}\Big)\overline{V\Big(\frac{\ell_2}{X}\Big)} &= \mathbf{1}_{(d,n_0/d)=1}\sum_{\ell\in\mathbb{N}} V\Big(\frac{\ell}{X}\Big) \sum_{\substack{k\in\mathbb{Z} \\ (k,d)=1}} \overline{V\Big(\frac{\ell+kn_0/d}{X}\Big)} \\ &= \mathbf{1}_{(d,n_0/d)=1}\Big\{\frac{\varphi(d)}{n_0}|X\widehat{V}(0)|^2 + O_{\varepsilon,D}(d^\varepsilon X)\Big\}. \end{aligned} \tag{54}$$

Note that the property $\rho(p^\nu) = \rho(p) \in \{0,2\}$ (for $p\nmid 2D$, $\nu\geq 1$) implies

$$\rho([n_1,n_2]) \sum_{\substack{d\mid(n_1,n_2) \\ (d,(n_1,n_2)/d)=1}} 1 = \rho([n_1,n_2])2^{\omega((n_1,n_2))} = \rho(n_1)\rho(n_2). \tag{55}$$

We insert the estimate (54) with $X = x/(Mt)$ in the RHS of (53) (recall that the additional hypothesis $M \leq x$ was justified at section 0.3.1). The factors $\varphi(d)$ compensate, and the relation (55) allows us to deduce $P_1 = P_1' + O(x^{1+\varepsilon})$, with

$$P_1' = \frac{|x\widehat{V}(0)|^2}{M} C_D \sum\sum_{N<n_1,n_2\leq 2N} b_{n_1}\overline{b_{n_2}\frac{\rho(n_1)\rho(n_2)}{n_1n_2}}A(n_1n_2)\int_{\mathbb{R}} t^{-2}f(t)\mathrm{d}t.$$

We then have $P_1' = |x\widehat{V}(0)|^2 P_3$, and finally

$$S_1 = |x\widehat{V}(0)|^2 P_3 + O_{\varepsilon,D}\Big(x^{1+\varepsilon} + x^{2+\varepsilon}\Big(\frac{N^2}{M}\Big)^{\frac{3}{2}-\theta}\Big). \tag{56}$$

Inserting the estimates (50), (51), and (56) in (49), we obtain the desired bound (2). This concludes the proof of Proposition 3

0.4. **Proof of Corollary 2.** In this section, we deduce Corollary 2 from the bound (8). We follow the arguments and notations of sections 4 and 5 de [DI82]. We consider

$$R_H(x,P,D) = \sum_{D<d\leq 2D}\lambda_d \sum_{0<|h|\leq H}\sum_{m\equiv 0 \pmod d}\frac{C(m)\log m}{m}\sum_{\nu^2\equiv D \pmod m}\widehat{b}\Big(\frac{h}{m}\Big)\mathrm{e}\Big(-\frac{h\nu}{m}\Big),$$

where $D \leq x^{\frac{1}{2}}$, $P \in [x, x^2]$, $\eta > 0$ is arbitrary, $H = Px^{-1+\eta}$, b is a smooth function compactly supported in $[x, 2x]$, such that $\|b^{(j)}\|_\infty \ll_j x^{-j}$, C is a smooth function compactly supported in $[P, 4P]$, such that $\|C^{(j)}\|_\infty \ll P^{-j}$, and (λ_d) is a sequence of coefficients with $|\lambda_d| \leq 1$. We insert the definition

$$\frac{1}{m}\widehat{b}\Big(\frac{h}{m}\Big) = \int_{\mathbb{R}} \mathrm{e}(-ht)b(mt)\mathrm{d}t.$$

Let $M = P/D$ and $f_{d,t}(v) = C(Mvd)\log(Mvd)b(Mvdt)$. We obtain

$$|R_H(x,P,D)| \ll xP^{-1}\sup_{|t|\in[x/(4P),2x/P]}\sum_{D<d\leq 2D}\Big|\sum_{0<|h|\leq H}\mathrm{e}(th)\sum_m f_{d,t}\Big(\frac{m}{M}\Big)\sum_{\nu^2\equiv D \pmod m}\mathrm{e}\Big(-\frac{h\nu}{md}\Big)\Big|.$$

We have $\|f_{d,t}^{(j)}\|_\infty \ll_j 1$, $D \ll M$ and $H \ll MD$. We may therefore apply the bound (8) to each dyadic subsum $H_1 < h \leq 2H_1$, for $\frac{1}{2} \leq H_1 \leq H$. We obtain

$$\begin{aligned}R_H(x,P,D) &\ll x^{1+\varepsilon+O(\eta)}P^{-1}D\sup_{\frac{1}{2}\leq H_1\leq H} H_1\Big\{H_1 + M^{\frac{1}{2}} + H_1^{-\frac{1}{2}}D^{\frac{1}{2}-\theta}M^{\frac{1}{2}+\theta}\Big\}\\ &\ll x^{\varepsilon+O(\eta)}\Big\{x^{-1}DP + (DP)^{\frac{1}{2}} + x^{\frac{1}{2}}P^\theta D^{1-2\theta}\Big\}.\end{aligned}$$

This is $O(x^{1-\eta})$ if $D \leq x^{-K\eta}\min\{x^2P^{-1}, x^{1/(2-4\theta)}P^{-\theta/(1-2\theta)}\}$ and K is a sufficiently large absolute constant. This bound on D, in conjunction with the arguments of section 8 of [DI82], yields the announced result.

0.5. **Proof of Lemma 6.**

Proof. Write $\sigma \equiv \left(\begin{smallmatrix} u & * \\ v & r\end{smallmatrix}\right)$ with $r \in \mathbb{Z}$. The classes $\Gamma_0(qd)\backslash\Gamma$ are in bijection with $\mathbb{P}^1(\mathbb{Z}/qd\mathbb{Z})$, the correspondance being given by $\sigma \mapsto [v : r]$. The condition $q|C(\sigma Q)$ then corresponds to $q|Q(v,r)$.

The relation $q|C(\sigma Q) = Q(v,r)$ implies $v|Q(v,r)^2$. However, we have the congruence $Q(v,r) \equiv Q(0,1)r^2 \pmod v$ and $(r,v) = 1$, so that finally $v|Q(0,1)^2$.

The explicit expression of $\mathcal{C}(\infty,\mathfrak{a})$ and of $S_{\infty\mathfrak{a}}(h,n;\gamma)$ is an elementary computation similar to section 2.2 of Deshouillers-Iwaniec [DI83]. We omit the details. The bound (23) is deduced using the triangle inequality, and noting that the condition $\alpha\delta \equiv u \pmod{vm}$ determines $\alpha \pmod{vm/(u,m)}$.

For the proof of (24), we use the Chinese remainder theorem. Let p be a prime number, and let

$$p^\mu \| m, \quad p^\lambda \| u, \quad p^\nu \| v \quad p^{\nu'} \| v', \quad p^\Delta \| d.$$

Our hypotheses $(v, u) = (v', m) = 1$ then imply

$$\mu > 0 \Rightarrow \nu' = 0, \quad \lambda > 0 \Rightarrow \nu = 0, \quad \Delta \le \max\{\nu, \nu'\}.$$

The Chinese remainder theorem shows that if suffices to prove the bounds

$$S_p(h) := \sum_{\substack{\alpha \pmod{p^{\nu+\mu}} \\ \delta \pmod{p^{\lambda+\mu+\max\{\nu,\nu'\}}} \\ \delta \equiv m \pmod{p^{\lambda+\nu'}} \\ (\delta - m, p^{\lambda+\mu}) = p^\lambda \\ \alpha\delta \equiv u \pmod{p^{\nu+\mu}}}} \chi_p(\delta) \mathrm{e}\Big(\frac{h\alpha}{p^{\nu+\mu}}\Big) \ll_Q (p^\Delta h, p^\mu), \tag{57}$$

where χ_p is a character modulo p^Δ. The change of variables $\delta \leftarrow m + \delta p^{\lambda+\nu'}$ transforms the LHS into

$$S_p(h) = \sum_{\substack{\alpha \pmod{p^{\nu+\mu}} \\ \delta \pmod{p^{\mu+\max\{\nu-\nu',0\}}} \\ (\delta, p^\mu) = 1 \\ \alpha(m+\delta p^{\lambda+\nu'}) \equiv u \pmod{p^{\nu+\mu}}}} \chi_p(m + \delta p^{\lambda+\nu'}) \mathrm{e}\Big(\frac{h\alpha}{p^{\nu+\mu}}\Big).$$

We first deal with the case $\mu \le \max\{\lambda, \nu\}$, taking the trivial bound

$$S_p(h) \ll_Q 1,$$

which follows from the fact that $u, v \ll_Q 1$.

Suppose then that $\mu > \max\{\lambda, \nu\} \ge 0$, in particular, $\nu' = 0$. Consider first the case $\nu = 0$, which implies $\Delta = 0$, so that the character is trivial and the sum simplies to

$$S_p(h) = \sum_{\substack{\alpha \pmod{p^\mu} \\ (\alpha, p) = 1}} \mathrm{e}\Big(\frac{h\alpha}{p^\mu}\Big) \sum_{\substack{\delta \pmod{p^\mu} \\ \alpha\delta \equiv u/p^\lambda \pmod{p^{\mu-\lambda}}}} 1 = p^\lambda c_{p^\mu}(h),$$

where $c_r(h) = \sum_{b \pmod r, (b,r)=1} \mathrm{e}(hb/r)$ is the Ramanujan sum. We obtain

$$|S_p(h)| \le p^\lambda (h, p^\mu).$$

Consider then the case $\nu > 0$. This implies $\lambda = 0$ and $\Delta \le \nu$, and so

$$S_p(h) = \sum_{\substack{\alpha \pmod{p^{\nu+\mu}} \\ \delta \pmod{p^{\nu+\mu}} \\ (\delta, p) = 1 \\ \alpha(m+\delta) \equiv u \pmod{p^{\nu+\mu}}}} \chi(m + \delta) \mathrm{e}\Big(\frac{h\alpha}{p^{\nu+\mu}}\Big) = \chi(u) \sum_{\substack{\alpha \pmod{p^{\nu+\mu}} \\ (\alpha, p) = 1}} \chi(\overline{\alpha}) \mathrm{e}\Big(\frac{h\alpha}{p^{\nu+\mu}}\Big)$$

which is a Gauss sum (*c.f.* [IK04, lemme 3.2]). We therefore have

$$|S_p(h)| \le 2(p^\Delta h, p^{\nu+\mu}).$$

We obtain in any case the bound (57), which concludes the proof. □

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