LIMIT LAWS FOR RATIONAL CONTINUED FRACTIONS AND VALUE DISTRIBUTION OF QUANTUM MODULAR FORMS

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Abstract. We study the limiting distributions of Birkhoff sums of a large class of cost functions (observables) evaluated along orbits, under the Gauss map, of rational numbers in $(0,1]$ ordered by denominators. We show convergence to a stable law in a general setting, by proving an estimate with power-saving error term for the associated characteristic function. This extends results of Baladi and Vallée on Gaussian behaviour for costs of moderate growth.

We apply our result to obtain the limiting distribution of values of several key examples of quantum modular forms. We show that central values of the Esterman function (L function of the divisor function twisted by an additive character) tend to have a Gaussian distribution, with a large variance. We give a dynamical, "trace formula free" proof that central modular symbols associated with a holomorphic cusp form for $SL(2,\mathbb{Z})$ have a Gaussian distribution. We also recover a result of Vardi on the convergence of Dedekind sums to a Cauchy law, using dynamical methods.

1. Introduction

1.1. **Continued fraction coefficients.** The present paper is concerned with dynamical properties of the Gauss map $T : x \mapsto \{1/x\}$, defined on $(0,1]$, and their application to the statistical behaviour of number-theoretic objects coming from the theory of modular forms. Iterated at an irrational number $x \in (0,1), x \notin \mathbb{Q}$, the Gauss map yields the continued fraction expansion

$$x = [0; a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}},$$

where $a_j = \lfloor 1/T^{j-1}(x) \rfloor$. Basic results in ergodic theory [CFS82, p. 174] describe the almost-sure average behaviour of iterates of T on irrational numbers in $(0,1)$. Define

$$S_N(f,x) := \sum_{j=1}^{N} f(a_j(x)).$$

Obtaining precise limiting behaviour through the study of spectral properties of transfer operators associated with T is an important theme in smooth dynamics and stationary Markov chains [Doe40, For40, ITM50, Wir74, RE83, Bro96, AD01]. We refer in particular to [Bro96] and to the introduction of [AD01] for an extensive historical account and references. A prominent example is given by [Bro96, Theorem 8.1]: if $f : [0,1] \to \mathbb{R}$ is of bounded variations and not of the shape $c+g-g\circ T$, for some function g of bounded variations and some constant $c \in \mathbb{R}$, then uniformly for $t \in \mathbb{R}$,

$$\mathbb{P}\Big(\frac{S_N(f,x) - \mu_f N}{\sigma_f \sqrt{N}} \le v\Big) = \Phi(v) + O\Big(\frac{1}{\sqrt{N}}\Big), \qquad \Phi(v) := \int_{-\infty}^{v} \frac{\mathrm{e}^{-t^2}\,\mathrm{d}t}{\sqrt{2\pi}}$$

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where $x \in (0,1)$ is chosen uniformly according to the Lebesgue measure. The implied constant may depend only on f. This relies on the spectral analysis of perturbations of the Gauss-Kuzmin-Wirsing transfer operator

$$\mathbb{H}[f](x) = \sum_{n\geq 1} \frac{1}{(n+x)^2} f\Big(\frac{1}{n+x}\Big).$$

Among maps of the interval, the Gauss map has been particularly studied because of its link with the analysis [May76, Pol86, May91b, MV12] of geodesic flows on the surface $SL(2,\mathbb{Z})\backslash\mathfrak{h}$ (where $\mathfrak{h}$ is the upper-half plane).

The above pertains the case of irrational $x \in (0,1)$. When $x \in \mathbb{Q} \cap (0,1)$, the continued fraction algorithm terminates, and letting $r = r(x)$ denote the least non-negative integer such that $T^r(x) = 0$, we have

$$x = [0; a_1, \dots, a_r]$$

with a_j defined as above. Since there are only a finite number of terms, we may similarly define

$$S(f,x) = \sum_{j=1}^{r(x)} f(a_j(x)),$$

and ask for the distribution of the values $S(f,x)$ as x ranges in the rational numbers in $(0,1]$. Note that now, the length of the sum is also varying with x. For every $Q \geq 1$, we endow the space

$$\Omega_Q := \{x = a/q,\ 1 \leq a \leq q \leq Q, (a,q) = 1\} \subset \mathbb{Q} \cap (0,1]$$

with the uniform probability measure. In [Val00], Vallée has shown that the expectations of $S(f,x)$ satisfy

$$\mathbb{E}_Q(S(f,x)) := |\Omega_Q|^{-1} \sum_{x\in\Omega_Q} S(f,x) = \mu_f \log Q + \nu_f + O(Q^{-\delta})$$

for a large class of functions f. Here the number $\delta > 0$ is absolute and the implied constant may depend on f. The numbers μ_f, ν_f depend only on f, and in fact

$$\mu_f = \frac{12\log 2}{\pi^2} \int_0^1 f(\lfloor 1/x \rfloor)\xi(x)\,\mathrm{d}x, \qquad \xi(x) := \frac{1}{(1+x)\log 2}.$$

An important point is that this question was studied within the framework of dynamical methods. The argument uses the construction of a suitable generating series involving the quasi-inverse $(\mathrm{Id} - \mathbb{H}_\tau)^{-1}$ of twisted transfer operators

$$\mathbb{H}_\tau[f] = \sum_{n\geq 1} \frac{1}{(n+x)^{2+i\tau}} f\Big(\frac{1}{n+x}\Big), \tag{1.1}$$

for arbitrary large $\tau \in \mathbb{R}$ (by contrast with the continuous setting, which involves perturbations of a single fixed operator). This construction crucially relies on the fact that the denominator $q(x)$ of x can be detected by means of the Birkhoff sum $\log q(x) = -\sum_{j=1}^{r(x)} \log(T^{j-1}(x))$. Earlier approaches [Hei69, Dix70], restricted to $f = 1$, involved number-theoretic methods based on bounds on algebraic exponential sums.

The approach of [Val00] was developed by Baladi and Vallée [BV05a], who proved that under the size condition $f(n) = O(\log(2n))$, the Laplace transform satisfies the "quasi-powers expansion"

$$\mathbb{E}_Q(\mathrm{e}^{wS(f,x)}) = \exp\{U(w)\log Q + V(w) + O(Q^{-\delta})\} \tag{1.2}$$

uniformly for w in some complex neighborhood of 0; here the holomorphic functions U, V, the number $\delta > 0$ and the implied constant may depend on f. This has a number of consequences in terms of the limiting distribution, and among them, an effective central limit theorem: if f

is real and $f(n) = O(\log n)$ for $n \geq 2$, then for some numbers $\mu_f \in \mathbb{R}$, $\sigma_f > 0$ and all $t \in \mathbb{R}$ we have

$$\mathbb{P}_Q\Big(\frac{x - \mu_f \log Q}{\sigma(\log Q)^{1/2}} \leq t\Big) = |\Omega_Q|^{-1}\Big|\Big\{x \in \Omega_Q,\ \frac{x - \mu_f \log Q}{\sigma(\log Q)^{1/2}} \leq t\Big\}\Big|$$
$$= \int_{-\infty}^{t} \frac{\mathrm{e}^{-v^2/2}\,\mathrm{d}v}{\sqrt{2\pi}} + O\Big(\frac{1}{\sqrt{\log Q}}\Big) \qquad (t \in \mathbb{R}).$$

An explicit formula for the mean value was given above; as far as we know, there is no closed form expression for σ_f in general, although it can be approximated in polynomial time (see [CGL11]). The power-saving error term in (1.2) depends on proving a pole-free strip for the quasi-inverse of the twists (1.1), which can be viewed as a "Riemann hypothesis" for the generating function of interest[1]. The approach of [BV05a] extends seminal work of Dolgopyat [Dol98] to the case of an expanding interval map with an infinite partition (see also [BV05b]).

1.2. **Limit laws.** In the present work we extend the results of Baladi and Vallée [BV05a] in two directions, motivated by several arithmetic questions from the theory of quantum modular forms [Zag10]. In the first direction, we consider cost functions $\phi : (0, 1] \to \mathbb{R}$ which depend on the iterates $b_j(x) := T^{j-1}(x)$, rather than the coefficients $a_j(x) = \lfloor 1/b_j(x)\rfloor$. Our main object of study is

$$S_\phi(x) := \sum_{j=1}^{r(x)} \phi(T^{j-1}(x)). \tag{1.3}$$

The setting of [BV05a] corresponds to taking $\phi(x) = f(\lfloor 1/x\rfloor)$. We will require that our cost functions can be extended to a Hölder continuous function, with some uniform exponent, on each interval $[\frac{1}{n+1}, \frac{1}{n}]$.

The second direction we wish to consider is cost functions $\phi(x)$ having a possibly divergent first or second moment, say $\phi(x) = x^{-1/2}$, or $\phi(x) = \lfloor 1/x\rfloor$, with the consequence that the limit law will be stable, but not necessarily Gaussian anymore [BV05a, p. 384]. This is a well-known theme in the theory of sums of independent random variables; see Chapter VI of [Lév25], or Chapters VI.1 and XVII.5-6 of [Fel71]. The corresponding phenomenon for sums of continued fractions coefficients in the *continuous setting* has been investigated by elementary means by Lévy [Lév52] (see also [Hei87]), and later by transfer operator methods [GH88, GLJ93, Sze09, AD01]. In fact, this falls into the general "countable Markov-Gibbs" framework of [AD01], where it is refered to as the "distributional limit" problem. A large part of later work has focused on non-uniformly hyperbolic maps; we refer to the survey [Gou15] and the references therein.

We investigate the corresponding question in the *discrete setting.* Our main result is the evaluation, with effective, power-saving error terms, of the characteristic function

$$\mathbb{E}_Q(\mathrm{e}^{itS_\phi(x)})$$

for t in a real neighborhood of 0, under hypotheses which essentially reduce to the boundedness of some positive absolute moment $\int_0^1 |\phi(x)|^{\alpha_0}\,\mathrm{d}x$ ($\alpha_0 > 0$). Our result involves the integral

$$\mathfrak{I}_\phi(t) := \int_0^1 (\mathrm{e}^{it\phi(x)} - 1)\xi(x)\,\mathrm{d}x \qquad (t \in \mathbb{R}).$$

Theorem 1.1. *Let $\varepsilon, \kappa_0, \alpha_0, \lambda_0 > 0$ with $\kappa_0 \leq 1$, and let $\phi : (0, 1] \to \mathbb{R}$ be a function such that for all $n \geq 1$, the function ϕ extends to a κ_0-Hölder continuous function on the interval $[\frac{1}{n+1}, \frac{1}{n}]$, and moreover*

$$\sum_{n\geq 1} \frac{1}{n^2}\Big(\sup_{x\in[\frac{1}{n+1},\frac{1}{n}]} |\phi(x)|^{\alpha_0} + \sup_{x,y\in[\frac{1}{n+1},\frac{1}{n}]} \frac{|\phi(x) - \phi(y)|^{\lambda_0}}{|x - y|^{\kappa_0\lambda_0}}\Big) < \infty. \tag{1.4}$$

[1]The same set of idea imply a zero-free strip for the Selberg zeta function of the full modular group [Nau05].

Then there exists $t_0, \delta > 0$ such that for $|t| \leq t_0$, the function $x \in \mathbb{Q} \cap (0,1] \mapsto S_\phi(x)$ defined in (1.3) *satisfies*

$$\mathbb{E}_Q\big(\mathrm{e}^{itS_\phi(x)}\big) = \exp\Big\{\Big(\frac{12\log 2}{\pi^2}\mathfrak{I}_\phi(t) + O(t^2 + |t|^{2\alpha_0-\varepsilon})\Big)\log Q + O(Q^{-\delta} + |t| + |t|^{\alpha_0-\varepsilon})\Big\}. \tag{1.5}$$

If moreover $\alpha_0 > 1$, then there exists a constant $C_\phi \in \mathbb{R}$ such that

$$\mathbb{E}_Q\big(\mathrm{e}^{itS_\phi(x)}\big) = \exp\Big\{\Big(\frac{12\log 2}{\pi^2}\mathfrak{I}_\phi(t) + C_\phi t^2 + O(t^3 + |t|^{1+\alpha_0-\varepsilon})\Big)\log Q + O(Q^{-\delta} + |t| + |t|^{\alpha_0-\varepsilon})\Big\}. \tag{1.6}$$

The value of δ, t_0 and the implied constant depend at most on $\alpha_0, \kappa_0, \lambda_0, \varepsilon$ and an upper-bound for the left-hand side of (1.4).

Remark. – It is important to note that the actual values of κ_0 and λ_0 only affect the statement up to the value of t_0, δ and the implied constants. In particular, if ϕ is $\mathcal{C}^1$ on each interval $[\frac{1}{n+1}, \frac{1}{n}]$ and if there exists $C \geq 1$ such that $|\phi'(x)| = O(x^{-C})$ for $x \in (0,1]$, then the second part of (1.4) is satisfied with $\kappa_0 = 1$ and any $\lambda_0 < 1/C$.
– Whenever $\phi(x) \asymp \phi(y)$ for all $x, y \in (\frac{1}{n+1}, \frac{1}{n})$ uniformly in n, the first part of hypothesis (1.4) is equivalent to $\int_0^1 |\phi(x)|^{\alpha_0}\,\mathrm{d}x < \infty$.
– In typical applications, the quantity $\mathfrak{I}_\phi(t)$ can be evaluated by standard methods, some of which we recall in Appendix A.
– The results of [BV05a] are stated in a formalism which includes the Gauss map as a special case. The most crucial assumption, at least as far as one is interested in power-saving error terms, is the "uniform non-integrability" assumption [BV05a, p.357] (see [Mor15] for a qualitative result, not using the UNI condition). In the present work, we do not use any more specific properties of the Gauss map; with suitable modifications, the arguments presented here apply to the centered and odd Euclidean algorithms [BV05a, Fig. 1] as well.

We will consider a slight generalization (Theorem 2.1 below), in order to deal with the map $x \mapsto -1/x$ (which arises naturally in modular forms theory), and in order to study joint distributions.

The main feature of Theorem 1.1 we use is the relative weakness of the hypotheses on ϕ. This is important in the applications we will discuss shortly, where little is known on ϕ besides the regularity properties, and rough bounds on the Hölder norms. To obtain this uniformity, we systematically use Hölder spaces, not only because of the regularity of ϕ, but also in order to dampen the oscillations of $\mathrm{e}^{it\phi}$ (see (3.3) below). This is the main reason why arbitrarily small values of λ_0 are admissible for Theorem 1.1.

The shape of the asymptotic estimate (1.5)–(1.6) is characteristic of infinitely divisible distribution [Fel71, Chapter XVII.2]. Along with an evaluation of $\mathfrak{I}_\phi(t)$ and the Berry-Esseen inequality [Ten15, Theorem II.7.16], [FS09, Theorem IX.5], we will deduce, in all of our applications, the convergence in law of a suitable renormalization of the random variable $S_\phi(x)$, $x \in \Omega_Q$, as $Q \to \infty$. When $\alpha_0 > 2$, Theorem 1.1 implies the following central limit theorem (CLT), which recovers in particular [BV05a, Theorem 3.(a)] (see also [BH08, Remark 1.3]).

Theorem 1.2. *Assume that the bound* (1.4) *holds for some $\alpha_0 > 2$. Suppose that ϕ is not of the form $c\log + f - f \circ T$ for a function $f : [0,1] \to \mathbb{R}$ and $c \in \mathbb{R}$. Then, for some $\sigma > 0$ and*

$$\mu = \frac{12}{\pi^2}\int_0^1 \frac{\phi(x)\,\mathrm{d}x}{1+x}, \tag{1.7}$$

we have

$$\mathbb{P}_Q\Big(\frac{S_\phi(x) - \mu\log Q}{\sigma\sqrt{\log Q}} \leq v\Big) = \Phi(v) + O\Big(\frac{1}{\sqrt{\log Q}} + \frac{1}{(\log Q)^{\alpha_0/2-1-\varepsilon}}\Big) \tag{1.8}$$

uniformly in $v \in \mathbb{R}$.

A variation on the argument shows that the milder hypothesis $\alpha_0 = 2$ implies the estimate (1.8) with a qualitative error term $o(1)$ as $Q \to \infty$.

For any $k \in \mathbb{N}_{>0}$, under the condition $\alpha_0 > k$, a variation on the arguments also leads to an estimate with power-saving error term for $\mathbb{E}_Q(S_\phi(x)^k)$, *cf.* the remark after Theorem 2.1.

Remark. A generalization of a different kind of Baladi-Vallée's results, for maps associated to a reduction algorithm in congruence subgroups, has very recently and independently been obtained by Lee and Sun [LS19].

1.3. **Large moments of continued fraction coefficients.** For all $\lambda \geq 0$, $x \in \mathbb{Q} \cap (0,1)$, let

$$M_\lambda(x) := a_1^\lambda + \cdots + a_r^\lambda, \qquad (x = [0; a_1, \ldots, a_r], a_r > 1). \tag{1.9}$$

When $\lambda < 1/2$, Theorem 1.2, with $\alpha_0 = 1/\lambda$ and

$$\phi_\lambda(x) := \lfloor 1/x \rfloor^\lambda,$$

implies that M_λ satisfies a CLT with mean and variance of order $\log Q$. We therefore focus on the case $\lambda \geq 1/2$.

For $0 < \alpha < 2$, define

$$c_\alpha = \Big(\frac{\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})}{\pi^2/12} \Big)^{1/\alpha} \tag{1.10}$$

and by continuity $c_1 = \frac{6}{\pi}$. Let

$$g_\alpha(x) := \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-itx - (c_\alpha t)^\alpha (1 - i\,\mathrm{sgn}(t)\tan(\frac{\pi\alpha}{2}))}\,\mathrm{d}x, & (\alpha \neq 1) \\ \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-itx - c_1 t(1 - i\frac{2}{\pi}\,\mathrm{sgn}(t)\log|t|)}\,\mathrm{d}x, & (\alpha = 1) \end{cases}$$

be the probability distribution function of a stable law $S_\alpha(c_\alpha, 1, 0)$ (see [ST94]), and

$$G_\alpha(v) := \int_{-\infty}^{v} g_\alpha(x)\,\mathrm{d}x.$$

Theorem 1.3. *Let $\lambda \geq 1/2$ and $v \in \mathbb{R}$, and for $\lambda < 1$ define $\mu_\lambda = \frac{12}{\pi^2}\sum_{n\geq 1} n^\lambda \log(\frac{(n+1)^2}{n(n+2)})$.*

(1) If $\lambda = 1/2$, then with $\sigma = (\pi^2/6)^{-1/2}$, we have

$$\mathbb{P}_Q\Big(\frac{M_{1/2}(x) - \mu_{1/2}\log Q}{\sigma\sqrt{\log Q \log\log Q}} \leq v \Big) = \Phi(v) + O\Big(\frac{1}{(\log\log Q)^{1-\varepsilon}} \Big). \tag{1.11}$$

(2) If $1/2 < \lambda < 1$, then

$$\mathbb{P}_Q\Big(\frac{M_\lambda(x) - \mu_\lambda \log Q}{(\log Q)^\lambda} \leq v \Big) = G_{1/\lambda}(v) + O\Big(\frac{1}{(\log\log Q)^{1-\varepsilon}} \Big). \tag{1.12}$$

(3) If $\lambda = 1$, then letting γ_0 denote the Euler constant,

$$\mathbb{P}_Q\Big(\frac{M_1(x)}{\log Q} - \frac{\log\log Q - \gamma_0}{\pi^2/12} \leq v \Big) = G_1(v) + O\Big(\frac{1}{(\log Q)^{1-\varepsilon}} \Big). \tag{1.13}$$

(4) If $\lambda > 1$, then

$$\mathbb{P}_Q\Big(\frac{M_\lambda(x)}{(\log Q)^\lambda} \leq v \Big) = G_{1/\lambda}(v) + O\Big(\frac{1}{(\log\log Q)^{1-\varepsilon}} \Big). \tag{1.14}$$

In all four cases the implied constant depends at most on ε and λ.

Except for (1.12), we expect the error terms to be optimal up to an exponent ε. The estimate (1.14) is in accordance with results on the statistical distribution of $\max_{1\leq j\leq r} a_j$ [Hen91, CV11].

The estimate (1.13) plainly implies the following statement, which answers a question in Section 10 of [FVV].

Corollary 1.4. *For $Q \geq 3$ and $\varepsilon > 0$, we have*
$$\Big|\sum_{j=1}^{r} a_j - \frac{12}{\pi^2}\log Q \log\log Q\Big| \leq \log Q (\log\log Q)^{\frac{1}{2}+\varepsilon}$$
for all fractions $\frac{a}{q} = [0; a_1, \dots, a_r]$ with $1 \leq a \leq q \leq Q$ and $(a,q)=1$, with at most $O_\varepsilon(\frac{Q^2}{\sqrt{\log\log Q}})$ exceptions.

This shows that the typical time complexity of the "substractive" algorithm for the GCD is $(1+o(1))\frac{12}{\pi^2}(\log Q)\log\log Q$ on pairs of coprime numbers at most Q, as $Q \to \infty$. This is in sharp contrast with the average time complexity, which is $(1+o(1))\frac{6}{\pi^2}(\log Q)^2$ (the latter is known even with a single average over numerators, see [YK75]).

1.4. **Value distribution of quantum modular forms.** Our main motivating application of Theorem 1.1 is the value distribution of quantum modular forms, introduced by Zagier [Zag10] (see [Zag99] for early examples). They were later studied by numerous mathematicians, in connection with Chern-Simons theory and quantum knot invariants [DGLZ09, Gar18, DG18, GZ], mock modular forms and partition theory [BOPR12, FOR13, Fol14, CLR16], Maass wave forms and the cohomology of covers of the modular surface [Bru07, BLZ15, CL16], cotangent sums appearing in the Nyman-Beurling criterion for the Riemann hypothesis [BC13].

Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a subgroup, acting on $\mathbb{Q} \cup \{\infty\}$ by Möbius transformations, and define for $k \in \mathbb{Z}$ the weight-k "slash operator" on functions $f : \mathbb{Q} \cup \{\infty\} \to \mathbb{C}$, by
$$f|_k\gamma(z) := (cz+d)^{-k} f(\gamma z) \qquad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$
A quantum modular form of weight[2] $k \in \mathbb{Z}$ for with respect to a Γ is a function $f\colon \mathbb{Q} \to \mathbb{C}$ such that for all $\gamma \in \Gamma$ the function
$$h_\gamma : \mathbb{Q} \smallsetminus \{\gamma^{-1}\infty\} \to \mathbb{C}, \qquad h_\gamma(x) := f(x) - (f|_k\gamma)(x)$$
has "some property of continuity or analyticity"[3] with respect to the real topology. The case $\Gamma = SL(2,\mathbb{Z})$ is particularly interesting in our context, since $\Gamma = SL(2,\mathbb{Z}) = \langle U, S\rangle$ with
$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
The functions f we will consider are periodic, so that $h_U \equiv 0$. We are therefore interested in 1-periodic functions $f : \mathbb{Q} \to \mathbb{C}$ such that the function
$$h_S(x) = f(x) - x^{-k} f(-1/x), \qquad x \in \mathbb{Q} \smallsetminus \{0\}, \tag{1.15}$$
extends to a "nice" (*e.g.*, continuous) function on $\mathbb{R} \smallsetminus \{0\}$. To simplify the presentation we assume, in this subsection only, that $2 \mid k$ and h_S is even; the general case can be studied by minor modifications to the argument.

A natural question is to ask how the values $f(x)$ are distributed when x varies in the set Ω_Q, which we recall consists in rationals in $(0,1]$ of denominators at most Q, as $Q \to \infty$. By iterations of the Gauss map (Euclid algorithm), together with the periodicity of f and the definition (1.15), we obtain the sum of iterates
$$q(x)^{-k} f(x) = f(0) + \sum_{j=0}^{r(x)-1} u_j(x)^{-k} h_S(T^j(x)), \tag{1.16}$$
where $u_j(x) := \big(\prod_{\ell=j}^{r(x)-1} T^\ell(x)\big)^{-1} \in \mathbb{N}$ and $q(x)$ is the reduced denominator of $x \in \mathbb{Q} \cap (0,1]$.

We first make a comment on the case $k > 0$. The right-hand side here is then expressed more conveniently in terms of the mirror element $\overline{x} \in \mathbb{Q} \cap (0,1]$ defined by $\overline{x} =$

[2] One can also consider non integral weight, but here we will actually be mostly interested in the case $k = 0$.

[3] The definition is purposely left a bit vague in [Zag10].

$[0; a_{r(x)}, \dots, a_1]$ (mod 1) (we have $\overline{x} = \overline{a}/q$ if $r(x)$ is odd, $x = a/q$ and $a\overline{a} \equiv 1 \pmod q$). Changing j to $r(x) - j$ yields

$$q(x)^{-k} f(\overline{x}) = f(0) + \sum_{j=1}^{r(x)} v_j(x)^{-k} h_S\Big(\frac{v_{j-1}(x)}{v_j(x)}\Big),$$

where the partial denominators $v_j(x)$ are defined for $x \in \mathbb{R}$ by

$$v_{-1}(x) = 0, \quad v_0(x) = 1, \quad v_{j+2}(x) = v_{j+1}(x)\Big\lfloor 1/T^{j+1}(x)\Big\rfloor + v_j(x).$$

Under a mild growth hypothesis, such as $h_S(|x|) = O(x^{-k} \log x)$ as $x \to 0$, say, the techniques used in [Bet15] can be used to show that the series

$$F(x) := \sum_{j=1}^{\infty} v_j(x)^{-k} h_S\Big(\frac{v_{j-1}(x)}{v_j(x)}\Big)$$

converges for Lebesgue-almost all $x \in \mathbb{R}$ and that the set $\{q(x)^{-k} f(x), x \in \Omega_Q\}$ becomes distributed as $Q \to \infty$ with respect to the push-forward $F_*(\mathrm{d}\nu)$ of the Lebesgue measure ν on $[0, 1]$. When f arises from an object which satisfies a suitable functional equation (special values of L-functions, say), then analogous considerations apply also for $k < 0$.

The case $k = 0$ is rather different and often arithmetically more interesting, and is in the scope of Theorem 1.1 with the function h_S playing the rôle of ϕ, since (1.16) yields the Birkhoff sum

$$f(x) = f(0) + \sum_{j=0}^{r(x)-1} h_S(T^j(x)).$$

Whenever it can be applied, Theorem 1.1 will guarantee that the values $\{f(x), x \in \Omega_Q\}$ become distributed as $Q \to \infty$ according to some stable law, characterized by the regularity and growth properties of h_S around 0 through an analysis of the integral $\mathfrak{I}_{h_S}(t)$. We describe the conclusions in detail in the following sections.

1.4.1. *Modular symbols.* Modular symbols can be defined in various ways (see [Man09]), for instance, as elements of the space of linear functionals on $S_k(\Gamma_0(N))$, the space of cusp form of weight $k \equiv 0 \pmod 2$ and level $N \geq 1$, spanned by the Shimura integrals

$$f(z) \mapsto \langle x \rangle_{f,m} := \frac{(2\pi i)^m}{(m-1)!} \int_x^{i\infty} f(z)(z - x)^{m-1}\, \mathrm{d}z =: \langle x \rangle_{f,m}^{+} + i\langle x \rangle_{f,m}^{-}$$

for any $1 \leq m \leq k - 1$, $x \in \mathbb{Q}$, and where $\langle x \rangle_{f,m}^{\pm} \in \mathbb{R}$. The modular symbol $\langle x \rangle_{f,m}$ is also (up to an explicit factor) the special value $L(f, x, m)$ of the analytic continuation of the L-function $L(f, x, s) := \sum_{n=1}^{\infty} a_n \mathrm{e}(nx) n^{-s}$, where $f(z) := \sum_{n \geq 1} a_n \mathrm{e}(nz)$ for $\mathrm{Im}(z) > 0$. Being at the intersection of the geometric, modular and arithmetic aspects of $\Gamma_0(N)$, modular symbols received a considerable amount of interest. For example, they can be used to compute modular forms and the homology of modular curves [Cre97].

Motivated by conjectures on the size of the algebraic rank of elliptic curves, Mazur, Rubin, and Stein [MR16, Ste15] have recently formulated a series of conjectures about the value distribution of $\langle x \rangle_{f,m}^{\pm}$ where f is a fixed form of weight $k = 2$ (and so $m = 1$) and $x = \frac{a}{q}$ varies among reduced fractions of denominator q, with $q \to \infty$. In particular, they predicted that

$$\Big\{\frac{\langle a/q \rangle_{f,1}^{\pm}}{\sqrt{\log q}}, 1 \leq a \leq q, (a, q) = 1\Big\}$$

becomes asymptotically distributed according to a suitably dilated normal law. Petridis and Risager [PR18] have recently shown that, with an extra average over q, the question can be solved by studying Eisenstein series twisted with modular symbols.

In the case where $k \geq 4$, there are $k - 1$ possible choices for m, but in fact only the case $m = \frac{k}{2}$ is truly interesting from the distributional point of view, following our discussion above.

Indeed, for $m \geq \frac{k}{2}+1$ the series defining $L(f,x,m)$ converges to a continuous function of $x \in \mathbb{R}$, and so the values $\langle x\rangle_{f,m}$ for $x \in \Omega_Q$ become distributed, as $Q \to \infty$, according to the measure $(L(f,\cdot,m))_*(\mathrm{d}\nu)$. The case $m \leq \frac{k}{2}-1$ reduces to the case $m \geq \frac{k}{2}+1$, since the functional equation for $s \mapsto L(f,x,s)$ relates $\langle \frac{a}{q}\rangle_{f,m}$ with $\langle \frac{\overline{a}}{q}\rangle_{f,k-m}$.

For level $N=1$ and $m=k/2$, the modular symbol $\langle x\rangle_{f,\frac{k}{2}}$ satisfies the reciprocity formula (1.15) with h_S being a bounded, Hölder-continuous function on $[0,1]$. Theorem 1.1 naturally leads to the following central limit theorem.

Corollary 1.5. *Let $k \geq 12$ be even, and $f \in S_k(SL(2,\mathbb{Z})) \smallsetminus \{0\}$ be fixed. Then for all $Q \geq 2$ and rectangle $\mathscr{R} \subset \mathbb{C}$, we have*

$$\mathbb{P}_Q\Big(\frac{\langle x\rangle_{f,\frac{k}{2}}}{\sigma_f\sqrt{\log Q}} \in \mathscr{R}\Big) = \int_{v_1+iv_2\in\mathscr{R}} \frac{\mathrm{e}^{-(v_1^2+v_2^2)/2}\,\mathrm{d}v_1\,\mathrm{d}v_2}{2\pi} + O\Big(\frac{1}{\sqrt{\log Q}}\Big), \tag{1.17}$$

with $\sigma_f^2 = \frac{3(4\pi)^k}{\pi\Gamma(k)}\|f\|_k^2$, where $\|f\|_k$ is the weight-k Petersson norm of f.

For example this result applies for $k=12$ with f being the discriminant modular form $\Delta(z)$. The error term is optimal and uniform with respect to $\mathscr{R}$.

Corollary 1.5, and in fact its generalization for all levels $N \geq 1$, has recently and independently been obtained by Nordentoft [Nor18], by the approach of [PR18].

In the recent preprint [LS19], Lee and Sun have independently obtained a proof of the main theorem of [PR18], for any level but for weight $k=2$, also by generalizing the methodology of Baladi-Vallée in a different direction. This is achieved by considering a certain twisted version of the Gauss map[4], keeping track at each iteration of a coset in $\Gamma\backslash SL_2(\mathbb{Z})$. On the other hand, as far as the analogy goes, the analogue in their setting of the cost function ϕ is constant; this is specific to the case $k=2$. By mixing the methods presented here with those of [LS19], it is plausible that the main results of [Nor18], for arbitrary weight and levels, could be recovered by dynamical methods.

The arguments of [PR18, Nor18] make use of the analytic continuation of Eisenstein series, and the location of eigenvalues of the Laplace operator acting on $L^2(\mathfrak{h}/SL(2,\mathbb{Z}))$, notably a spectral gap, obtained through the use of trace formulas. By contrast, the present work does not substantially use properties of $SL(2,\mathbb{Z})$ beyond the fact that it is a group.

A second application closely related to modular symbols is the distribution of values of the following rather elementarily-defined function. Given $k \geq 2$ an even integer and $D \in \mathbb{N}$, $D \equiv 0$ or 1 (mod 4) which is not a square, define

$$A_{D,k}(x) := \sum_{\substack{\mathrm{disc}(Q)=D \\ Q(x)>0>a_Q}} Q(x)^{k-1}, \qquad x \in \mathbb{R}$$

where the sum is over all quadratic functions $Q \in \mathbb{Z}[x]$ of discriminant D and leading term a_Q. As shown in [Zag99, Ben15], the sum is actually finite if $x \in \mathbb{Q}$, and converges for all $x \in \mathbb{R}$. For all $x \in \mathbb{Q}$, the value $A_{D,k}(x)$ is in fact essentially the modular symbol $\langle x\rangle^+_{f_{D,k},2k-1}$ associated with some cusp form $f_{D,k}$ of weight $2k$ [Zag75].

It is shown in [Zag99, eq. (55)] that the function $A_{D,k}$, extended to all $\mathbb{R}$, is differentiable $k-2$ times but not $k-1$ times if $k \geq 6$. Since the sum above is finite for $x \in \mathbb{Q}$ (cf. [Zag99, p.1147]), we may consider the naively regularized derivative

$$F_{D,k}(x) := \sum_{\substack{\mathrm{disc}(Q)=D \\ Q(\alpha)>0>a_Q}} \frac{d^{k-1}}{dy^{k-1}}(Q(y)^{k-1})\big|_{y=x} \qquad (x \in \mathbb{Q}).$$

[4]Another way to study levels $N>1$ would be by building an expanding map out of Atkin-Lehner homographies, but our attempts to construct such a map satisfying the UNI condition were not successful.

This is well-defined for $x \in \mathbb{Q}$, but not anymore for $x \notin \mathbb{Q}$: it defines a quantum modular form very closely related to the central modular symbol $\langle x \rangle_{f_{D,k},k}$. We obtain the following central limit theorem for the values of $F_{D,k}$.

Corollary 1.6. *Let $k \geq 6$ be even. For some $\sigma_{D,k} > 0$ and all $Q \geq 3$, we have*

$$\mathbb{P}_Q\Big(\frac{F_{D,k}(x)}{\sigma_{D,k}\sqrt{\log Q}} \leq v\Big) = \Phi(v) + O\Big(\frac{1}{\sqrt{\log Q}}\Big).$$

The constant $\sigma_{D,k}$ can be expressed in terms of the Petersson norm of a certain cusp form (see (8.10) below), which can be expressed in turn as a restricted double sum of a certain function over pairs of quadratic forms. We did not find a particularly simple form of the ensuing expression, therefore we refrain from carrying this out here.

1.4.2. *Central values of the Estermann function.* The original motivation of the present work is the Estermann function, defined by

$$D(s,x) := \sum_{n\geq 1} \frac{\tau(n)\mathrm{e}(nx)}{n^s}$$

for $\mathrm{Re}(s) > 1$, and extended by analytic continuation otherwise. Here $\tau(n) := \sum_{d|n} 1$. Like the modular symbols, the central values $D(\frac12, x)$ carry deep arithmetic information: the twisted second moment of Dirichlet L-functions satisfies

$$\begin{aligned} M(a,q) &:= \frac{1}{q^{1/2}} \sum_{\chi \pmod q} |L(\tfrac12,\chi)|^2\chi(a) \\ &= \mathrm{Re}\, D(\tfrac12, \tfrac{a}{q}) + \mathrm{Im}\, D(\tfrac12, \tfrac{a}{q}) + O(q^{-1/2}) \end{aligned} \tag{1.18}$$

for q prime and $q \nmid a$ (see [Bet16, Theorem 5]). From this expression, the fourth moment of Dirichlet L-functions $\sum_{\chi \pmod q} |L(\frac12,\chi)|^4$, whose full evaluation in [You11] is at the threshold of current techniques of analytic number theory, is essentially the second moment $\sum_{a \pmod q} |M(a,q)|^2$ (see [BFK$^+$17a, BFK$^+$17b] for further work on this topic). This fits in the general problem of understanding the distribution of central values of L-functions and their twists, which is a fundamental topic in analytic number theory [Sel92, CFK$^+$05, Sou09, Har13, RS15]. Up to now, essentially all known results have been obtained by the moments method.

One would like to interpret $D(\frac12, x)$ as a regularized modular symbol for the derivative $\frac{\partial}{\partial s}E_2(z,s)|_{s=1/2}$ of the Eisenstein series (see [Iwa02, chapter 3.5]), but since they are not cuspidal, we lack a suitable representation such as the Shimura integral to construct a corresponding multiplier system, and it is not clear how to adapt the approach of [PR18].

In [Bet16], it was shown that $D(\frac12, x)$ satisfies (1.15) with h_S being a $(\frac12 - \varepsilon)$-Hölder continuous function on $(0,1]$. By contrast with the case of modular symbols, now $h_S(x)$ is not bounded in the neighborhood of 0 and in fact $h_S(x) \sim \frac{1+i}{2}x^{-1/2}\log(1/x)$ as $x \to 0^+$. Theorem 1.1 applies and yields the following.

Corollary 1.7. *For all $\varepsilon > 0$, $Q \geq 3$, and all rectangle $\mathscr{R} \subset \mathbb{C}$, we have*

$$\mathbb{P}_Q\Big(\frac{D(\frac12,x)}{\sigma(\log Q)^{\frac12}(\log\log Q)^{\frac32}} \in \mathscr{R}\Big) = \int_{v_1+iv_2\in\mathscr{R}} \frac{\mathrm{e}^{-(v_1^2+v_2^2)/2}\,\mathrm{d}v_1\,\mathrm{d}v_2}{2\pi} + O\Big(\frac{1}{(\log\log Q)^{1-\varepsilon}}\Big) \tag{1.19}$$

where $\sigma = 1/\pi$.

Note the difference between the variances in (1.17) and (1.19). The phenomenon responsible for this difference (*i.e.* that h_S has a singularity at 0) also causes the moments method to be ineffective in this problem. All the integer moments of $D(\frac12, x)$ and $M(a,q)$ have recently been computed in [Bet]. They grow faster that what is suggested by Corollary 1.7: they are dominated by a negligible proportion of $x \in \Omega_Q$ with abnormally large continued fraction coefficients.

Another possible approach to Corollary 1.7 consists in obtaining a limiting distribution result for $\sum_{n\geq 1}\tau(n)\mathrm{e}(nz)$, where $z = x+iy$; $x\in[0,1]$ is chosen at random and $y\to 0^+$, and transfering these properties to its discrete counterpart $D(\frac12,x)\approx\sum_{n\leq q^2}\tau(n)\mathrm{e}(nx)n^{-1/2}$, for $x=a/q\in\mathbb{Q}$, by Fourier expansion. This method is employed for the incomplete Gauss sum in [DA14], based on [Mar99]. In our setting, however, connecting the continuous and the discrete averages raises several additional difficulties, among which the problem of dealing with a divergent second moment and of considering a central value rather than an object corresponding to an L-value off the line. It is unclear to us how to adapt this method to our case.

It would be interesting to obtain an analogous statement for the values $M(a,q)$, however the identity (1.18) as stated holds only for q prime; the corresponding identity for generic q involves an additional multiplicative convolution, and it is not clear how to adapt the dynamical method to account for it. However we still believe $M(a,q)$ is distributed according to a normal law.

Conjecture 1.8. *As $q\to\infty$ among primes, the values*

$$\Big\{\frac{M(a,q)}{(\log Q)^{1/2}(\log\log Q)^{3/2}},\ 1\leq a<q\Big\}$$

become distributed according to a suitably dilated centered Gaussian.

1.4.3. *Dedekind sums.* For all coprime integers a and q with $q\geq 1$, define the Dedekind sum

$$s\Big(\frac{a}{q}\Big) := \sum_{h=1}^{q-1}\Big(\Big(\frac{ha}{q}\Big)\Big)\Big(\Big(\frac{h}{q}\Big)\Big),\qquad ((x)) := \begin{cases}\{x\}-1/2 & (x\notin\mathbb{Z}),\\ 0 & (\text{otherwise}).\end{cases}$$

As a straightforward consequence of our work, we recover the following statement, which is the main result of [Var93].

Corollary 1.9. *Uniformly for $v\in\mathbb{R}$ and $Q\geq 2$, we have*

$$\mathbb{P}_Q\Big(\frac{s(x)}{\log Q}\leq\frac{v}{2\pi}\Big)=\frac{1}{\pi}\int_{-\infty}^{v}\frac{\mathrm{d}y}{1+y^2}+O\Big(\frac{1}{(\log Q)^{1-\varepsilon}}\Big). \tag{1.20}$$

This is easily deduced as a special case of our general results (Theorem 2.1 below). The argument in [Var93] strongly relies on the special form of the reciprocity formula for $s(a/q)$, which allows to reduce the problem to an analysis of twisted Poincaré series on the modular surface, which are in turn studied by means of the Kuznetsov trace formula. By contrast, as we have already mentioned, our arguments are essentially of a dynamical nature, and uses little arithmetic information beyond the group structure of $SL(2,\mathbb{Z})$.

1.4.4. *Kashaev invariants of the 4_1 knot.* In [Zag10], Zagier introduces the modularity conjecture on Kashaev invariants of knots. To a knot K and an integer $n\geq 2$, one can associate a Laurent polynomial $J_{K,n}(\mathbf{q})\in\mathbb{Z}[[\mathbf{q}]]$, called the n-colored Jones polynomial; see [Gar18, section 1]. If $\mathbf{q}\in\mathbb{C}$ is a root of unity of order $n\geq 1$, one can then define a function $J_{K,0}(\mathbf{q})$ as the value $J_{K,n}(\mathbf{q})$. In [MM01], this was shown to be equal to Galois orbits of he Kashaev invariants. In the case of the 4_1 knot (or "figure-eight" knot), the simplest hyperbolic knot, one has

$$J_{4_1,0}(\mathrm{e}(x)) := \sum_{m=0}^{\infty}|1-\mathbf{q}|^2\cdots|1-\mathbf{q}^m|^2,\qquad \mathbf{q}=\mathrm{e}(x),\ x\in\mathbb{Q}.$$

Note that for each given $x\in\mathbb{Q}$, the sum is finite. In this case, Zagier's modularity conjecture, stated precisely in [Zag10], predicts that $x\mapsto\log J_{4_1,0}(\mathrm{e}(x))$ satisfies a certain reciprocity formula of weight 0, in particular the difference

$$\log J_{4_1,0}(\mathrm{e}(-1/x))-\log J_{4_1,0}(\mathrm{e}(x)),$$

which is depicted in [Zag10, Fig. 4] is expected to behave "nicely" with respect to x, although not continuously. A proof of Zagier's conjecture for the 4_1 knot has been announced in [GZ]. In [BD], we obtained independently another proof, complemented by a reciprocity formula relative to a transformation of another kind (essentially the conjugation of the Gauss map by the map $a/q \mapsto \overline{a}/q$, where $a\overline{a} \equiv 1 \pmod q$). Using Theorem 1.1, we then deduced that

$$\log J_{4_1,0}(\mathrm{e}(x)) \sim C \log q \log\log q, \qquad C = \frac{\mathrm{Vol}(4_1)}{2\pi} \approx 0.323$$

for almost all $x \in \mathbb{Q} \cap (0,1]$ of denominator $q \leq Q$, as $Q \to \infty$. We also prove that a more precise convergence in law similar to (1.13) would follow if one can prove a weak form of continuity for the relevant period function h_S. We do not give more details here and refer the interested reader to [BD].

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2. Setting

In this section, we describe the slightly more general setting in which we will work. For a parameter $\kappa \in [0,1]$, a real interval I and a normed space X let $H^\kappa(I,X)$ denotes the set of functions $I \to X$ such that the semi-norm

$$\|f\|_{(\kappa)} := \sup_{x,y\in I, x\neq y} \frac{\|f(x)-f(y)\|}{|x-y|^\kappa} \tag{2.1}$$

is finite.

We let $\mathcal{H} := \{h : [0,1] \to \mathbb{R} \mid \exists n \geq 1 \text{ s.t. } h(x) = \frac{1}{n+x}\}$ be the inverse branches of the Gauss map, and $\mathcal{H}^\ell := \{h_1 \circ \cdots \circ h_\ell \mid h_j \in \mathcal{H}\}$.

We are given an integer $m \geq 1$, and m functions $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_m : [0,1] \to \mathbb{R}^d$. We extend this definition by periodicity, letting $\boldsymbol{\phi}_j := \boldsymbol{\phi}_{j \pmod m}$ for all $j \geq 1$, and we define a function on $\mathbb{Q} \cap (0,1]$ by letting $S(1) := 0$ and

$$S(x) := \sum_{j=1}^{r(x)} \boldsymbol{\phi}_j(T^{j-1}(x)), \qquad (x \in \mathbb{Q} \cap (0,1))$$

where

$$r(x) := \min\{j \geq 0, T^j(x) = 0\}.$$

For $x \in (0,1)$ with $r(x) \geq m$, we define

$$\boldsymbol{\phi}(x) := \sum_{j=1}^{m} \boldsymbol{\phi}_j(T^{j-1}(x)), \tag{2.2}$$

We make the following hypotheses:

(1) (κ_0-Hölder continuity) For each $n \geq 1$ and $1 \leq j \leq m$, the function $\boldsymbol{\phi}_j$ can be extended as a κ_0-Hölder continuous function on the interval $[\frac{1}{n+1}, \frac{1}{n}]$.

(2) (Norm α_0-th moment) We have

$$\sum_{h\in\mathcal{H}^m} |h'(0)| \Big\| \boldsymbol{\phi}|_{h(\mathcal{I})} \Big\|_\infty^{\alpha_0} < \infty. \tag{2.3}$$

(3) (Hölder λ_0-th moment) We have

$$\sum_{h\in\mathcal{H}^m} |h'(0)| \Big\| \boldsymbol{\phi}|_{h(\mathcal{I})} \Big\|_{(\kappa_0)}^{\lambda_0} < \infty. \tag{2.4}$$

For all $\boldsymbol{t} \in \mathbb{R}^d$, we denote $\|\boldsymbol{t}\| := \|\boldsymbol{t}\|_\infty$. Finally, let

$$\mathfrak{I}_{\boldsymbol{\phi}}(\boldsymbol{t}) := \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}(x)\rangle} - 1)\xi(x)\,\mathrm{d}x.$$

Theorem 2.1. *Let $\kappa_0, \alpha_0, \lambda_0 > 0$ be given with $\kappa_0, \lambda_0 \leq 1$, and $\boldsymbol{\phi} : (0,1] \to \mathbb{R}$ satisfying the conditions (2.3)–(2.4). Then there exists $t_0 > 0$, $\delta > 0$, and two functions $U, V : \{\boldsymbol{t} \in \mathbb{R}^d, \|\boldsymbol{t}\| \leq t_0\} \to \mathbb{C}$ such that for all $\boldsymbol{t} \in \mathbb{R}^d$ with $\|\boldsymbol{t}\| \leq t_0$, we have*

$$\mathbb{E}_Q(\mathrm{e}^{i\langle \boldsymbol{t}, S(x)\rangle}) = \exp\{U(\boldsymbol{t})\log Q + V(\boldsymbol{t}) + O(Q^{-\delta})\},$$

and

$$\begin{aligned} U(\boldsymbol{t}) &= \frac{12\log 2}{m\pi^2}\int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}(x)\rangle} - 1)\xi(x)\,\mathrm{d}x + O(\|\boldsymbol{t}\|^2 + \|\boldsymbol{t}\|^{2\alpha_0-\varepsilon}), \\ V(\boldsymbol{t}) &= O(\|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon}). \end{aligned} \tag{2.5}$$

If moreover $\alpha_0 > 1$, then there exists a real $d \times d$ matrix $C_{\boldsymbol{\phi}}$ such that

$$U(\boldsymbol{t}) = \frac{12\log 2}{m\pi^2}\int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}(x)\rangle} - 1)\xi(x)\,\mathrm{d}x + \boldsymbol{t}^T C_{\boldsymbol{\phi}}\boldsymbol{t} + O(\boldsymbol{t}^3 + |\boldsymbol{t}|^{1+\alpha_0-\varepsilon}) \tag{2.6}$$

where $\boldsymbol{t}$ is intepreted as a column vector, and $\boldsymbol{t}^T$ is the transpose. The dependence of δ and the implied constant on the parameters is as in Theorem 1.1.

Remark. If $\alpha > k$, by a more precise analysis of the arguments in Section 6, we can show that the functions U and V are of class $\mathcal{C}^k$, which leads to the k-th moment estimate remarked upon after Theorem 1.1.

Notations. For any function $f(s,t)$ of two real or complex variables, and all $k, \ell \geq 0$, we let $\partial_{k,\ell} f := \frac{\partial^{k+\ell}}{\partial s^k \partial t^\ell} f$ whenever this function is defined.

We recall that the semi-norm $\|f\|_{(\kappa)}$ is defined in (2.1). The Landau symbol $f = O(g)$ means that there is a constant $C \geq 0$ for which $|f| \leq Cg$ whenever f and g are defined. The notation $f \ll g$ means $f = O(g)$. If the constant depends on a parameter, say ε, this is indicated in subscript, *e.g.* $f = O_\varepsilon(g)$ or $f \ll_\varepsilon g$.

3. Lemmas

3.1. Hölder constants.

We compile here facts we will use on the Hölder norms $\|f\|_{(\kappa)}$ for $f \in H^\kappa(I, \mathbb{C})$.

(1) For $f, g \in H^\kappa$, we have

$$\|fg\|_{(\kappa)} \leq \|f\|_{(\kappa)}\|g\|_\infty + \|f\|_\infty\|g\|_{(\kappa)}. \tag{3.1}$$

(2) For $g \in H^1$ and $f \in H^\kappa(I)$, we have $f \circ g \in H^\kappa$ and

$$\|f \circ g\|_{(\kappa)} \leq \|g\|_{(1)}^\kappa \left\|f|_{g(I)}\right\|_{(\kappa)}. \tag{3.2}$$

(3) For $\lambda \in [\kappa, 1]$ and $f \in H^{\kappa/\lambda}$ real, we have

$$\left\|\mathrm{e}^{if}\right\|_{(\kappa)} \leq \|f\|_{(\kappa/\lambda)}^\lambda. \tag{3.3}$$

(4) For $0 < \kappa < \lambda$ and $f \in H^\lambda$, we have

$$\|f\|_{(\kappa)} \leq \|f\|_{(0)}^{1-\kappa/\lambda}\|f\|_{(\lambda)}^{\kappa/\lambda}. \tag{3.4}$$

3.2. **Oscillating integrals.** We will require the following analogue of van der Corput's lemma.

Lemma 3.1. *Let $\Delta, \kappa > 0$. Assume that $\Psi : [0,1] \to \mathbb{R}$ is $\mathcal{C}^1$ with $\Psi' \geq \Delta$ and that Ψ' is monotonous on I. Let $g \in H^\kappa$. Then*

$$\int_0^1 g(x)\mathrm{e}^{i\Psi(x)}\,\mathrm{d}x \ll \frac{\|g\|_\infty}{\Delta} + \frac{\|g\|_{(\kappa)}}{\Delta^\kappa}.$$

Proof. Let $I = [0,1]$. The lemma is obtained by combining the methods of the usual van der Corput Lemma ([Ste93], Proposition VIII.2, p. 332), and the bound on Fourier coefficients of a Hölder continuous function ([SS03], ex. 15, p. 92); we restrict to mentioning the main steps. We change variables and let

$$h(x) = \frac{g \circ \Psi^{-1}(x)}{\Psi' \circ \Psi^{-1}(x)}.$$

Let $R = (\Psi(I) \smallsetminus (\Psi(I) - \pi)) \cup (\Psi(I) \smallsetminus (\Psi(I) + \pi))$. We have

$$\begin{aligned}\int_0^1 g(x)\mathrm{e}^{i\Psi(x)}\,\mathrm{d}x &= \int_{\Psi(I)} h(x)\mathrm{e}^{ix}\,\mathrm{d}x \\ &= O\Big(\int_R \frac{\|g\|_\infty \,\mathrm{d}x}{\Psi' \circ \Psi^{-1}(x)}\Big) - \frac{1}{2}\int_{\Psi(I)\cap(\Psi(I)-\pi)} (h(x+\pi) - h(x))\mathrm{e}^{ix}\,\mathrm{d}x.\end{aligned}$$

Now, on the one hand,

$$\begin{aligned}h(x+\pi) - h(x) &= \frac{g\circ\Psi^{-1}(x+\pi) - g\circ\Psi^{-1}(x)}{\Psi'\circ\Psi^{-1}(x)} + (g\circ\Psi^{-1})(x+\pi)\Big(\frac{1}{\Psi'\circ\Psi^{-1}(x+\pi)} - \frac{1}{\Psi'\circ\Psi^{-1}(x)}\Big) \\ &\ll \frac{\|g\|_{(\kappa)}\|\Psi^{-1}\|_{(1)}^\kappa}{\Psi'\circ\Psi^{-1}(x)} + \|g\|_\infty\Big|\frac{1}{\Psi'\circ\Psi^{-1}(x+\pi)} - \frac{1}{\Psi'\circ\Psi^{-1}(x)}\Big|\end{aligned}$$

by (3.2) on the first term, and on the other hand,

$$\int_{\Psi(I)\cap(\Psi(I)-\pi)} \Big|\frac{1}{\Psi'\circ\Psi^{-1}(x+\pi)} - \frac{1}{\Psi'\circ\Psi^{-1}(x)}\Big|\,\mathrm{d}x \leq \int_R \frac{\mathrm{d}x}{\Psi'\circ\Psi^{-1}(x)} = O(1/\Delta)$$

by monotonicity. We conclude using $\|\Psi^{-1}\|_{(1)} \ll 1/\Delta$. □

4. Properties of the transfer operator

In this section and the following ones, all implied constants in the notations $O(\dots)$ and $\ll$ may depend on $\alpha_0, \kappa_0, \lambda_0$, m and an upper-bound for the values of (2.3)–(2.4). Additional dependences will be indicated in subscript.

Definition. Let

$$\kappa := \min(\tfrac{1}{3}, \tfrac{1}{2}\kappa_0\lambda_0), \tag{4.1}$$

where we recall that κ_0 is the Hölder exponent of $\boldsymbol{\phi}$ on each interval $(\frac{1}{n+1}, \frac{1}{n})$. For all $\boldsymbol{t} \in \mathbb{R}^d$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, define an operator $\mathbb{H}_{s,\boldsymbol{t}}^{(j)}$ acting on $H^\kappa([0,1],\mathbb{R})$ by

$$\begin{aligned}\mathbb{H}_{s,\boldsymbol{t}}^{(j)}[f](x) &= \sum_{n\geq 1} \frac{\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_j(1/(n+x))\rangle}}{(n+x)^s} f\Big(\frac{1}{n+x}\Big) \\ &= \sum_{h\in\mathcal{H}} \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_j\circ h\rangle}|h'(x)|^{s/2}(f\circ h)(x)\end{aligned}$$

with the notation of Section 2.

When $\boldsymbol{t} = (0,\dots,0)$, this is independent of j, in which case we drop the notation (j). We abbreviate further

$$\mathbb{H}_s := \mathbb{H}_{s,0}, \qquad \mathbb{H} := \mathbb{H}_{2,0}\,.$$

Define the norm and the semi-norm

$$\|f\|_0 := \|f/\xi\|_\infty, \qquad \|f\|_1 := \|f/\xi\|_{(\kappa)}.$$

Here we recall that $\xi(x)=\frac{1}{\log 2}\frac{1}{1+x}$. We equip $H^\kappa([0,1])$ with the norms $\|f\|_{1,\beta}:=\|f\|_0+\beta^{-\kappa}\|f\|_1$, where $\beta>0$.

For $\boldsymbol{t}\in\mathbb{R}^d$ and $s\in\mathbb{C}$, $\mathrm{Re}(s)>1$, and $0\le j\le m$, let

$$\Pi^{(j)}_{s,\boldsymbol{t}}:=\mathbb{H}^{(j)}_{s,\boldsymbol{t}}\cdots\mathbb{H}^{(1)}_{s,\boldsymbol{t}},$$

with $\Pi^{(0)}_{s,\boldsymbol{t}}=\mathrm{Id}$. In what follows, we will often abbreviate

$$\Pi_{s,\boldsymbol{t}}:=\Pi^{(m)}_{s,\boldsymbol{t}},$$

and to define

$$\mathfrak{T}(x):=\prod_{0\le j\le m-1}T^j(x),\qquad g_{s,\boldsymbol{t}}:=\mathrm{e}^{i\langle\boldsymbol{t},\boldsymbol{\phi}\rangle}\mathfrak{T}^{s-2}. \tag{4.2}$$

Note that $g_{s,\boldsymbol{t}}\circ h$ belongs to $H^{\kappa_0}([0,1])$ for all $h\in\mathcal{H}^m$. Also, we have

$$\Pi_{s,\boldsymbol{t}}:f\mapsto\mathbb{H}^m[g_{s,\boldsymbol{t}}f]. \tag{4.3}$$

4.1. **First properties.**

4.1.1. *Properties at the central point.* By [Bro96] (section 2.2, Proposition 4.1 and Theorem 4.2), along with the Ruelle-Perron-Frobenius theorem (see [May91a]), we have that the operator $\mathbb{H}_{2,0}$ acting on $H^\kappa([0,1])$ is quasi-compact. It has 1 as a simple eigenvalue, and no other eigenvalue of modulus ≥ 1. The projection associated with the eigenvalue 1 is given by

$$\mathbb{P}_{2,0}[f](x)=\Big(\int_{[0,1]}f\,\mathrm{d}\nu\Big)\xi(x),$$

where ν is the Lebesgue measure. Since $\Pi^{(m)}_{2,0}=\mathbb{H}^m_{2,0}$, we obtain the existence of a linear operator $\mathbb{N}_{2,0}$ acting on H^κ, such that

$$\Pi^{(m)}_{2,0}=\mathbb{P}_{2,0}+\mathbb{N}_{2,0}, \tag{4.4}$$

and additionally the spectral radius of $\mathbb{N}_{2,0}$ satisfies $\mathrm{srd}(\mathbb{N}_{2,0})<1$ and $\mathbb{P}_{2,0}\mathbb{N}_{2,0}=\mathbb{N}_{2,0}\mathbb{P}_{2,0}=0$.

4.1.2. *Dominant spectral properties.* In the sequel, we will repeatedly use the following facts.

– The bounded distortion property [BV05a, eq. (3.1)]: for all $h\in\mathcal{H}^*$, we have $|h''|\ll|h'|$ with a uniform constant, in particular, independently of the depth of h. This implies that for all $x\in[0,1]$ and $h\in\mathcal{H}^*$,

$$|h'(x)|\asymp|h'(0)|.$$

– For all $q>\frac{1}{2}$, we have

$$\sum_{h\in\mathcal{H}}\|h'\|_\infty^q<\infty.$$

– The "contracting ratio" property of the inverse branches [BV05a, bound 2.4, fig. 1]: for some $\rho\in[0,1)$, all $\ell\in\mathbb{N}$ and $h\in\mathcal{H}^\ell$, we have the uniform bound

$$\|h\|_{(1)}\ll\rho^\ell. \tag{4.5}$$

Lemma 4.1. – *For* $\mathrm{Re}(s)=\sigma>1$*, we have*

$$\Pi_{s,\boldsymbol{t}}[f]\le\Pi_{\sigma,0}[|f|].$$

– *For all $\sigma>1$, there exists $A_\sigma>0$ such the map $\sigma\mapsto A_\sigma$ is Lipschitz-continuous and decreasing, $A_2=1$, and we have the bounds on operator norms*

$$\|\mathbb{H}_{\sigma,0}\|_0\le A_\sigma,\qquad\|\Pi_{\sigma,0}\|_0\le A_\sigma^m. \tag{4.6}$$

In particular, $A_\sigma\le\mathrm{e}^{O(2-\sigma)}$ for $\sigma\in(1,2]$.

– *For some* $\rho < 1$*, all* $k \in \mathbb{N}$ *and all* $f \in H^k$ *with* $f \geq 0$*, we have*

$$\left\|\Pi_{2,0}^k[f]\right\|_0 \ll \int_{[0,1]} f \, \mathrm{d}\nu + \rho^k \|f\|_0. \tag{4.7}$$

The implied constant is absolute.

Proof. – The first statement is trivial by the triangle inequality.
– By a direct computation, we have

$$\|\mathbb{H}_\sigma\|_0 \leq A_\sigma := \sup_{x\in[0,1]} \frac{1}{\xi(x)} \sum_{n\geq 1} \frac{1}{(n+x)^\sigma} \xi\Big(\frac{1}{n+x}\Big).$$

The properties we require of A_σ are readily verified.
– The third item follows from (4.4) with any fixed $\rho \in (\mathrm{srd}(\mathbb{N}_{2,0}), 1)$, by the Gelfand inequality. □

4.1.3. *Spectral gap at* $\boldsymbol{t} = 0$ *and* $\tau \neq 0$*.*

Lemma 4.2. *For* $\tau \neq 0$*, we have* $\|\mathbb{H}_{2+i\tau}\|_0 < 1$*, and so similarly for* $\Pi_{2+i\tau,0}$*.*

Proof. This is a well-studied phenomenon; see [PP90, Proposition 6.1]. Our exact statement for $\mathbb{H}_{2+i\tau}$ acting on a space of holomorphic functions is proved in [Val03, p. 476] (see also [Mor15, Proposition 3.2]), however an inspection of the proof shows that it actually holds for $\mathbb{H}_{2+i\tau}$ acting on $\mathcal{C}([0,1])$, and therefore also on H^κ. □

4.1.4. *Perturbation.*

Lemma 4.3. *For all* $\varepsilon > 0$*, there exists* $\delta, t_0 > 0$ *such that the following holds.*
– *For* $\sigma \geq 2-\delta$ *and* $\|\boldsymbol{t}\| \leq t_0$*, we have*

$$\|\Pi_{s,\boldsymbol{t}} - \Pi_{s,0}\|_0 \ll_\varepsilon \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon}. \tag{4.8}$$

– *For* $2-\delta < \sigma \leq 3$ *and all* $\tau \in \mathbb{R}$*, we have*

$$\|\Pi_{\sigma+i\tau,0} - \Pi_{2+i\tau,0}\|_0 \ll |\sigma - 2|. \tag{4.9}$$

– *For* $\tau_1, \tau_2 \in \mathbb{R}$*, we have*

$$\|\Pi_{2+i\tau_1,0} - \Pi_{2+i\tau_2,0}\|_0 \ll |\tau_1 - \tau_2|. \tag{4.10}$$

Proof. – We have

$$\begin{aligned}\|\Pi_{s,\boldsymbol{t}} - \Pi_{s,0}\|_0 &= \sup_{\|f\|_0=1} \|\mathbb{H}^m[(g_{s,\boldsymbol{t}} - g_{s,0})f]\|_0 \\ &\ll \sum_{h\in\mathcal{H}^m} |h'(0)| \|(g_{s,\boldsymbol{t}} - g_{s,0})\circ h\|_\infty.\end{aligned}$$

However, for all $x \in (0,1]$, we have $|g_{s,\boldsymbol{t}}(x) - g_{s,0}(x)| \ll |h'(0)|^{\sigma/2-1}(\|\boldsymbol{t}\|\|\boldsymbol{\phi}(x)\|)^\alpha$ with $\alpha = \min(1, \alpha_0 - \varepsilon)$. By Hölder's inequality, we deduce

$$\begin{aligned}\|\Pi_{s,\boldsymbol{t}} - \Pi_{s,0}\|_0 &\ll \|\boldsymbol{t}\|^\alpha \sum_{h\in\mathcal{H}^m} |h'(0)|^{\sigma/2} \|\boldsymbol{\phi}\circ h\|_\infty^\alpha \\ &\ll \|\boldsymbol{t}\|^\alpha \Big(\sum_{h\in\mathcal{H}^m} |h'(0)| \|\boldsymbol{\phi}\circ h\|_\infty^{\alpha_0}\Big)^{\alpha/\alpha_0} \Big(\sum_{h\in\mathcal{H}^m} |h'(0)|^q\Big)^{1-\alpha/\alpha_0}\end{aligned}$$

with $q = \frac{\sigma\alpha_0 - 2\alpha}{2(\alpha_0-\alpha)}$. Picking $\delta = \frac{1}{2}(1 - \frac{\alpha}{\alpha_0}) = O(\varepsilon)$ ensures that $q > \frac{1}{2}$, and with our hypothesis (2.3) we obtain that both sums above are bounded in terms of ε and $\boldsymbol{\phi}$ only. Therefore we have $\|\Pi_{s,\boldsymbol{t}} - \Pi_{s,0}\| \ll_\varepsilon \|\boldsymbol{t}\|^\alpha$ as claimed.

– Proceeding as above, we find

$$\|\Pi_{\sigma+i\tau,0}-\Pi_{2+i\tau,0}\|_0 \ll \sum_{h\in\mathcal{H}^m}|h'(0)|\Big\|(\mathfrak{T}^{\sigma-2}-1)\circ h\Big\|_\infty$$
$$\ll |\sigma-2|\sum_{h\in\mathcal{H}^m}(1+|\log|h'(0)||)|h'(0)|^{\max(\sigma/2,1)}.$$

For any $\sigma>1$, the last sum is finite, so that our statement follows for any fixed $\delta\in(0,1)$.

– Once again proceeding again as above, we have

$$\|\Pi_{2+i\tau_1,0}-\Pi_{2+i\tau_2,0}\|_0 \ll \sum_{h\in\mathcal{H}^m}|h'(0)|\Big\|(\mathfrak{T}^{i(\tau_1-\tau_2)}-1)\circ h\Big\|_\infty.$$

Letting $\tau=\tau_1-\tau_2$, we insert the inequality $\|(\mathfrak{T}^{i\tau}-1)\circ h\|_\infty \ll |\tau|(1+|\log|h'(0)||)$. The resulting sum over h being absolutely bounded, we deduce $\|\Pi_{2+i\tau_1,0}-\Pi_{2+i\tau_2,0}\|_0\ll|\tau|$ as required. □

4.1.5. *First estimate on* $\|\Pi_{s,\boldsymbol{t}}\|_1$. The following is a weak form of [BV05a, Lemma 2] (which is refered to, there, as a Lasota-Yorke type inequality).

Lemma 4.4. *For all* $\delta\in(0,\kappa)$*, there exists* $\rho\in[0,1)$ *such that for* $\sigma\geq 2-\delta$*,* $\boldsymbol{t}\in\mathbb{R}^d$ *with* $\|\boldsymbol{t}\|\leq 1$*, and* $\tau\in\mathbb{R}$*, we have*

$$\|\Pi_{s,\boldsymbol{t}}[f]\|_1\leq O(1+|s|^\kappa)\|f\|_0+\rho\|f\|_1,$$

Proof. Let $f\in H^\kappa$. We write

$$\tfrac{1}{\xi}\Pi_{s,\boldsymbol{t}}[f]=\sum_{h\in\mathcal{H}^m}\tfrac{1}{\xi}(\xi\circ h)|h'|^{s/2}\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\circ h\rangle}(\tfrac{f}{\xi}\circ h).$$

Splitting as a sum of differences, we obtain

$$\Big\|\tfrac{1}{\xi}\Pi_{s,\boldsymbol{t}}[f]\Big\|_{(\kappa)}\leq\|f\|_0\sum_{h\in\mathcal{H}^m}\Big\|\tfrac{1}{\xi}(\xi\circ h)|h'|^{s/2}\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\circ h\rangle}\Big\|_{(\kappa)}+\rho_\sigma\|f\|_1,$$

where

$$\rho_\sigma:=\Bigg\|\sum_{h\in\mathcal{H}^m}\tfrac{1}{\xi}(\xi\circ h)|h'|^{\sigma/2}w_h\Bigg\|_\infty,\qquad w_h(x)=\sup_{0\leq y\leq 1}\left|\frac{h(x)-h(y)}{x-y}\right|^\kappa=|h'(x)h'(0)|^{\kappa/2}.$$

This last equality follows from the fact that each $h\in\mathcal{H}^*$ is a homography associated with an element of $GL_2(\mathbb{Z})$ with non-negative entries. Since $|h'(0)|\leq 1$ by the chain rule, we deduce

$$\sum_{h\in\mathcal{H}^m}\tfrac{1}{\xi}(\xi\circ h)|h'|^{\sigma/2}w_h\leq\sum_{h\in\mathcal{H}^m}\tfrac{1}{\xi}(\xi\circ h)|h'||h'|^{(\sigma-2+\kappa)/2}.$$

Note that $\sigma-2+\kappa\geq\kappa-\delta>0$ by hypothesis, and we have $|h'|\leq 1$ by the chain rule. Moreover, for any value of m, we may find at least one $h\in\mathcal{H}^m$ with $\|h'\|_\infty<1$, *e.g.* by composing repeatedly $t\mapsto\frac{1}{2+t}$. Since $\frac{1}{\xi}(\xi\circ h)|h'|>0$, we deduce

$$\rho_{\kappa-\delta}<\Bigg\|\sum_{h\in\mathcal{H}^m}\tfrac{1}{\xi}(\xi\circ h)|h'|\Bigg\|_\infty=1.$$

Next, by using the rules (3.2), (3.3), (3.4), the bounded distortion property $|h''|\ll|h'|$, and simple computations, we obtain successively

$$\Big\|\tfrac{1}{\xi}\Big\|_{(\kappa)}\ll 1,\qquad\qquad \Big\||h'|^{s/2}\Big\|_{(\kappa)}\ll|s|^\kappa\|h'\|^{\sigma/2},$$
$$\|\xi\circ h\|_{(\kappa)}\leq\|h'\|_\infty^\kappa,\qquad\qquad \Big\|\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\circ h\rangle}\Big\|_{(\kappa)}\ll\|h'\|_\infty^\kappa\|\boldsymbol{t}\|^{\kappa/\kappa_0}\|\boldsymbol{\phi}\|_{(\kappa_0)}^{\kappa/\kappa_0}.$$

In the last line, we used the definition (4.1). Grouping these bounds using (3.1), we deduce

$$\sum_{h\in\mathcal{H}^m} \left\| \tfrac{1}{\xi}(\xi\circ h)|h'|^{s/2}\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\circ h\rangle}\right\|_{(\kappa)} \ll 1+|s|^\kappa + \sum_{h\in\mathcal{H}^m} \|h'\|_\infty^{\sigma/2+\kappa}\|\boldsymbol{\phi}\|_{(\kappa_0)}^{\kappa/\kappa_0}$$
$$\ll 1+|s|^\kappa$$

since $\sigma/2+\kappa\geq 1+\kappa/2\geq 1$, $\|\boldsymbol{\phi}\|_{(\kappa_0)}^{\kappa/\kappa_0}\leq 1+\|\boldsymbol{\phi}\|_{(\kappa_0)}^{\lambda_0}$ (by the definition (4.1)), and by our hypothesis (2.4). ☐

5. Meromorphic continuation

Following [Val00], define the generating series

$$\mathfrak{S}(\boldsymbol{t},s) := \sum_{x\in\mathbb{Q}\cap(0,1]} q(x)^{-s}\exp(i\langle \boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle),$$

where $q(x)$ is the reduced denominator of x.

Lemma 5.1. *For* $\mathrm{Re}(s)>1$ *and* $\boldsymbol{t}\in\mathbb{R}^d$*, we have*

$$\mathfrak{S}(\boldsymbol{t},s) = (\Pi_{s,\boldsymbol{t}}^{(0)}+\Pi_{s,\boldsymbol{t}}^{(1)}+\dots+\Pi_{s,\boldsymbol{t}}^{(m-1)})(\mathrm{Id}-\Pi_{s,\boldsymbol{t}})^{-1}[\mathbf{1}](1).$$

Proof. This is a straightforward extension of the computations in [BV05a, p. 341]. ☐

The aim of this section is to show the meromorphic continuation of $\mathfrak{S}(s,t)$ to a half-plane $\mathrm{Re}(s)\geq 2-\delta$.

5.1. Small height.

Lemma 5.2. *There exists* $\delta,\tau_0,t_0>0$*, such that for all* $\sigma,\tau,t\in\mathbb{R}$ *with* $|\sigma-2|\leq\delta$*,* $|\tau|\leq\tau_0$ *and* $\|\boldsymbol{t}\|\leq t_0$*, the operator* $\Pi_{s,\boldsymbol{t}}$ *acting on* $(H^\kappa,\|\cdot\|_{1,1})$ *is quasi-compact, and for some* $\lambda(s,\boldsymbol{t})\in\mathbb{C}$*, we have*

$$\Pi_{s,\boldsymbol{t}} = \lambda(s,\boldsymbol{t})\mathbb{P}_{s,\boldsymbol{t}}+\mathbb{N}_{s,\boldsymbol{t}}$$

where $\mathbb{P}_{s,\boldsymbol{t}}$ *is of rank 1,* $\mathbb{P}_{s,\boldsymbol{t}}\mathbb{N}_{s,\boldsymbol{t}}=\mathbb{N}_{s,\boldsymbol{t}}\mathbb{P}_{s,\boldsymbol{t}}=0$*,* $\mathbb{P}_{s,\boldsymbol{t}}^2=\mathbb{P}_{s,\boldsymbol{t}}$*, and* $\mathrm{srd}(\mathbb{N}_{s,\boldsymbol{t}})<1-\delta$*. Moreover, for each such fixed* $\boldsymbol{t}$*, the operators* $\Pi_{s,\boldsymbol{t}}$*,* $\mathbb{P}_{s,\boldsymbol{t}}$*,* $\mathbb{N}_{s,\boldsymbol{t}}$ *and the eigenvalue* $\lambda(s,\boldsymbol{t})$ *depend analytically on* s*.*

Proof. This is a direct application of *e.g.* Theorem 2.3 of [Klo]; see also chapter IV.3 of [Kat95], and [BV05a, p.342]. ☐

5.2. Moderate height.

Lemma 5.3. *For all* $\tau_0,\tau_1>0$ *with* $\tau_0<\tau_1$*, there exists* $\delta,t_0>0$ *such that for all* $t,\sigma,\tau\in\mathbb{R}$ *with* $\|\boldsymbol{t}\|\leq t_0$*,* $\sigma\geq 2-\delta$ *and* $\tau_0\leq|\tau|\leq\tau_1$*, we have*

$$\|\Pi_{s,\boldsymbol{t}}\|_0\leq 1-\delta.$$

Proof. By (4.10) and the triangle inequality, the map $\tau\mapsto\|\Pi_{2+i\tau,0}\|_0$ is continuous. By Lemma 4.2, we deduce that for some number $\eta>0$, depending on τ_0,τ_1, we have $\|\Pi_{2+i\tau,0}\|_0\leq 1-\eta$. By the perturbation bounds (4.8) and (4.9), we may pick δ,t_0 small enough so that $\|\Pi_{s,\boldsymbol{t}}-\Pi_{2+i\tau,0}\|_0\leq\eta/2$, and our claim follows. ☐

5.3. Large height.

Lemma 5.4. *For some constants* $\delta,\tau_1,C>0$*, whenever* $\sigma\geq 2-\delta$*,* $\boldsymbol{t}\in\mathbb{R}^d$ *with* $\|\boldsymbol{t}\|\leq 1$*, and* $|\tau|\geq\tau_1$*, the operator* $\Pi_{s,\boldsymbol{t}}$ *acting on* H^κ *has spectral radius* $\mathrm{srd}(\Pi_{s,\boldsymbol{t}})<1$*, and*

$$\sum_{j\geq 0}\|\Pi_{s,\boldsymbol{t}}^j\|_{1,\tau}\ll|\tau|^{C|\sigma-2|}\log|\tau|.$$

For large values of τ, we adapt the arguments of Dolgopyat and Baladi-Vallée with two modifications: we work with Hölder-continuous functions, rather than $\mathcal{C}^1$, and the cost function is not assumed to be constant on each interval of the partition.

5.3.1. *Sums over branches.* We will require two estimates involving sums over inverse branches on T. Define, as in [BV05a, eq. (3.10)],

$$\Delta(h_1, h_2) := \inf_{x\in[0,1]} \left|\frac{h_1''(x)}{h_1'(x)} - \frac{h_2''(x)}{h_2'(x)}\right|,$$

Note that by the bounded distortion property, there exists $\Delta_+ \geq 1$ such that

$$\Delta(h_1, h_2) \leq \Delta_+ \tag{5.1}$$

for all $h_1, h_2 \in \mathcal{H}^*$. The following property is the statement that condition UNI.(a) of Baladi-Vallée [BV05a] holds for the Gauss map.

Lemma 5.5. *For some absolute constant $\rho \in [0, 1)$, uniformly in $n \in \mathbb{N}$, $h_1 \in \mathcal{H}^n$ and $u \in [0, \Delta_+]$ we have*

$$S(u) := \sum_{\substack{h_2\in\mathcal{H}^n \\ \Delta(h_1,h_2)\leq u}} |h_2'(0)| \ll \rho^n + u.$$

Proof. See Lemmas 1 and 6 of [BV05a]; the main point is the construction of a dual dynamical system (Section 3.4 of [BV05a]) which encodes naturally the quantity $\Delta(h_1, h_2)$, and satisfies the dominant spectral bound (4.7). The dual map of the Gauss map is in fact the Gauss map. □

Lemma 5.6. *Under the assumption* (2.4)*, uniformly for all $n \in \mathbb{N}_{>0}, 0 \leq j \leq n$ with $n-j \geq m$, we have*

$$\sum_{h\in\mathcal{H}^n} |h'(0)| \left\| \boldsymbol{\phi}|_{T^j\circ h(\mathcal{I})} \right\|_{(\kappa_0)}^{\lambda_0} \ll 1.$$

Proof. We decompose $h = h_1 \circ h_2 \circ h_3$, where $h_1 \in \mathcal{H}^j$, $h_2 \in \mathcal{H}^m$ and $h_3 \in \mathcal{H}^{n-j-m}$. We have

$$|h'(0)| \ll |h_1'(0)h_2'(0)h_3'(0)|, \qquad \left\| \boldsymbol{\phi}|_{T^j\circ h(\mathcal{I})} \right\|_{(\kappa_0)} \leq \left\| \boldsymbol{\phi}|_{h_2(\mathcal{I})} \right\|_{(\kappa_0)},$$

so that

$$\sum_{h\in\mathcal{H}^n} |h'(0)| \left\| \boldsymbol{\phi}|_{T^j\circ h(\mathcal{I})} \right\|_{(\kappa_0)}^{\lambda_0} \ll \Big(\sum_{h_1\in\mathcal{H}^j} |h_1'(0)| \Big) \Big(\sum_{h_2\in\mathcal{H}^m} |h_2'(0)| \left\| \boldsymbol{\phi}_{h_2(\mathcal{I})} \right\|_{(\kappa_0)}^{\lambda_0} \Big) \Big(\sum_{h_3\in\mathcal{H}^{n-j-m}} |h_3'(0)| \Big).$$

The sums over h_1 and h_3 are uniformly bounded by (4.6). The sum over h_2 is finite by our hypothesis (2.4). □

5.3.2. *Bound on the L^2 norm.*

Lemma 5.7. *For some $\delta, t_0 > 0$ and $\rho \in [0, 1)$, whenever $|\sigma - 2| \leq \delta$, $|\tau| \geq 1$, $\|\boldsymbol{t}\| \leq t_0$ and $\ell \in \mathbb{N}$, we have*

$$\Big(\int_{[0,1]} \left|\Pi_{s,\boldsymbol{t}}^{\ell}[f]\right|^2 \mathrm{d}\nu \Big)^{1/2} \ll A_{2\sigma-2}^{m\ell/2} \big((|\tau|^{-\kappa/2} + \rho^{m\ell/4}) \|f\|_0 + \rho^{\kappa m\ell/2} |\tau|^{-\kappa/2} \|f\|_1 \big).$$

Proof. Changing f to $\overline{f}$, $\boldsymbol{t}$ to $-\boldsymbol{t}$ and taking conjugates if necessary, we may assume that $\tau \geq 0$. Define, for all $\ell \in \mathbb{N}_{>0}$, $\boldsymbol{\psi}_\ell := \sum_{0\leq j<\ell} \boldsymbol{\phi} \circ T^{mj}$, so that

$$\Pi_{s,\boldsymbol{t}}^{\ell}[f] = \sum_{h\in\mathcal{H}^{m\ell}} \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\psi}_\ell\circ h\rangle} |h'|^{s/2} (f \circ h).$$

We note that for all $h \in \mathcal{H}^{m\ell}$, by (3.2) and (3.3), we have

$$\begin{aligned} \left\| \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\psi}_\ell\circ h\rangle} \right\|_{(\kappa)} &\leq \sum_{0\leq j<\ell} \left\| T^{mj} \circ h \right\|_{(1)}^{\kappa} \left\| \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle}|_{T^{mj}\circ h(\mathcal{I})} \right\|_{(\kappa)} \\ &\ll \sum_{0\leq j<\ell} \rho^{m(\ell-j)\kappa} \left\| \boldsymbol{\phi}|_{T^{mj}\circ h(\mathcal{I})} \right\|_{(\kappa_0)}^{\kappa/\kappa_0}. \end{aligned} \tag{5.2}$$

For $h_1, h_2 \in \mathcal{H}^{m\ell}$, let

$$g_{h_1,h_2} := \mathrm{e}^{i\langle \boldsymbol{t}, \psi_\ell\circ h_1 - \psi_\ell\circ h_2\rangle} |h_1' h_2'|^{\sigma/2} (f\circ h_1)\overline{(f\circ h_2)}.$$

This defines a function in H^κ. Expanding the square, we find

$$\int_{[0,1]} \left|\Pi^\ell_{s,\boldsymbol{t}}[f]\right|^2 \mathrm{d}\nu = \sum_{h_1,h_2\in\mathcal{H}^{m\ell}} I(h_1,h_2), \qquad I(h_1,h_2) := \int_0^1 g_{h_1,h_2}(x)\left|\frac{h_1'(x)}{h_2'(x)}\right|^{i\tau/2}\mathrm{d}x.$$

We have, for all $h_1, h_2 \in \mathcal{H}^{m\ell}$, the trivial bound

$$|I(h_1,h_2)| \ll \|g_{h_1,h_2}\|_\infty. \tag{5.3}$$

On the other hand, for all $h_1, h_2 \in \mathcal{H}^{m\ell}$ satisfying $\Delta(h_1,h_2) > 0$, we have from Lemma 3.1 the bound

$$|I(h_1,h_2)| \ll \frac{\|g_{h_1,h_2}\|_\infty}{|\tau|\Delta(h_1,h_2)} + \frac{\|g_{h_1,h_2}\|_{(\kappa)}}{(|\tau|\Delta(h_1,h_2))^\kappa}, \tag{5.4}$$

The norms are bounded, using (4.5), (3.1), (3.2) and (5.2), by

$$\begin{aligned} \|g_{h_1,h_2}\|_\infty &\ll \|f\|_\infty^2 |h_1'(0)h_2'(0)|^{\sigma/2}, \\ \|g_{h_1,h_2}\|_{(\kappa)} &\ll \|f\|_\infty |h_1'(0)h_2'(0)|^{\sigma/2}\{(1 + \sum_{h\in\{h_1,h_2\}}\sum_{0\le j<\ell} \rho^{m(\ell-j)\kappa}\left\|\phi|_{T^{mj}\circ h(\mathcal{I})}\right\|_{(\kappa_0)}^{\kappa/\kappa_0})\|f\|_\infty \\ &\qquad + \rho^{\kappa m\ell}\|f\|_{(\kappa)}\}. \end{aligned} \tag{5.5}$$

We write $\|f\|_{(\kappa)}\|f\|_\infty \ll \|f\|_{(\kappa)}^2 + \|f\|_\infty^2 \ll \|f/\xi\|_{(\kappa)}^2 + \|f/\xi\|_\infty^2$, which implies the variant

$$\|g_{h_1,h_2}\|_{(\kappa)} \ll |h_1'(0)h_2'(0)|^{\sigma/2}\Big\{\Big(1 + \sum_{h\in\{h_1,h_2\}}\sum_{0\le j<\ell} \rho^{m(\ell-j)\kappa}\left\|\phi|_{T^{mj}\circ h(\mathcal{I})}\right\|_{(\kappa_0)}^{\kappa/\kappa_0}\Big)\|f\|_0^2 + \rho^{\kappa m\ell}\|f\|_1^2\Big\}. \tag{5.6}$$

Next, for all $u \in [0, \Delta_+]$ (where we recall (5.1)), we have uniformly

$$\begin{aligned} K(u) := \max_{0\le j<\ell} &\sum_{\substack{h_1,h_2\in\mathcal{H}^{m\ell}\\ \Delta(h_1,h_2)\le u}} |h_1'(0)h_2'(0)|^{\sigma/2}(1 + \left\|\phi|_{T^{mj}\circ h_1(\mathcal{I})}\right\|_{(\kappa_0)}^{\lambda_0/2}) \\ \ll \max_{0\le j<\ell} &\Big(\sum_{h_1,h_2\in\mathcal{H}^{m\ell}} |h_1'(0)|(1 + \left\|\phi|_{T^{mj}\circ h_1(\mathcal{I})}\right\|_{(\kappa_0)}^{\lambda_0})|h_2'(0)|^{\sigma-1}\Big)^{1/2} \\ &\times \Big(\sum_{\substack{h_1,h_2\in\mathcal{H}^{m\ell}\\ \Delta(h_1,h_2)\le u}} |h_1'(0)|^{\sigma-1}|h_2'(0)|\Big)^{1/2} \\ \ll A_{2\sigma-2}^{m\ell}&(\rho^{m\ell/2} + u^{1/2}) \end{aligned}$$

by Lemmas 5.5 and 5.6. Let $\eta \in (0,1]$ be a parameter. We insert the bounds (5.5) and (5.6) in (5.3), (5.4), and we sum over (h_1,h_2). When $\Delta(h_1,h_2) \le \eta$, we use the trivial bound (5.3), otherwise we use (5.4). Using our bond on $K(u)$ above, the symmetry $h_1 \leftrightarrow h_2$, the fact that $\kappa/\kappa_0 \leq \lambda_0/2$, and partial summation, we find

$$\begin{aligned} A_{2\sigma-2}^{-m\ell} \sum_{h_1,h_2\in\mathcal{H}^{m\ell}} I(h_1,h_2) &\ll \|f\|_0^2\Big(K(\eta) + \frac{K(\Delta_+)}{\tau^\kappa} + \int_\eta^{\Delta_+} \Big(\frac{1}{\tau u} + \frac{\kappa}{(\tau u)^\kappa}\Big)\frac{K(u)\,\mathrm{d}u}{u}\Big) \\ &\quad + \frac{\rho^{\kappa m\ell}\|f\|_1^2}{\tau^\kappa}\Big(K(\Delta_+) + \kappa\int_\eta^{\Delta_+}\frac{K(u)\,\mathrm{d}u}{u^{\kappa+1}}\Big) \\ &\ll \|f\|_0^2\Big((\rho^{m\ell/2}+\eta^{1/2})\Big(1+\frac{1}{\tau\eta}+\frac{1}{(\tau\eta)^\kappa}\Big)+\frac{1}{\tau^\kappa}\Big) \\ &\quad + \|f\|_1^2\rho^{\kappa m\ell}\Big(\frac{1}{\tau^\kappa}+\frac{\rho^{m\ell/2}+\eta^{1/2}}{(\tau\eta)^\kappa}\Big). \end{aligned}$$

Choosing $\eta = 1/\tau$, we obtain

$$\Big(\int_{[0,1]}\Big|\Pi_{s,\boldsymbol{t}}^\ell[f]\Big|^2\,\mathrm{d}\nu\Big)^{1/2} \ll A_{2\sigma-2}^{m\ell}((\tau^{-\kappa/2}+\rho^{m\ell/4})\|f\|_0 + \tau^{-\kappa/2}\rho^{\kappa m\ell/2}\|f\|_1)$$

as claimed. ☐

5.3.3. *Bound on the L^∞ norm.* Next, we transfer the L^2 bound relative to the invariant measure into an L^∞ bound, following ideas of Dolgopyat [Dol98] adapted to this context by Baladi and Vallée [BV05a].

Lemma 5.8. *For some $\rho \in [0,1)$ and $c_0, \delta, \tau_0 > 0$, depending on η and κ at most, whenever*

$$\sigma \geq 2-\delta, \qquad \tau \geq \tau_0, \qquad \|\boldsymbol{t}\| \leq 1,$$

then letting $n = \lfloor c_0 \log\tau \rfloor$, we have

$$\Big\|\Pi_{s,\boldsymbol{t}}^n[f]\Big\|_0 \leq \rho^n \|f\|_{1,\tau}.$$

Proof. Using the Cauchy–Shwarz inequality, as in [BV05a, Lemma 1], for all $x \in [0,1]$ and $f \in H^\kappa$, we have

$$\begin{aligned} \frac{\Big|\Pi_{s,\boldsymbol{t}}^k[f](x)\Big|}{\xi(x)} &\leq \Big(\sum_{h\in\mathcal{H}^{mk}} |h'(x)|^{\sigma-1}\Big)^{1/2}\Big(\sum_{h\in\mathcal{H}^{mk}} |h'(x)|(|f|^2\circ h)(x)\Big)^{1/2} \\ &\ll A_{2\sigma-2}^{mk/2}\Big(\int_0^1 |f|^2\,\mathrm{d}\nu + \rho_1^{mk}\|f\|_0^2\Big)^{1/2} \end{aligned}$$

for some $\rho_1 \in [0,1)$ independent of k. We use this with f replaced by $\Pi_{s,\boldsymbol{t}}^\ell[f]$, with k,ℓ being any choice with $k+\ell = n$ and $\ell = k + O(1)$. By Lemma 5.7 and the bound (4.6), we deduce that for all small enough $\delta \geq 0$, if $\sigma \geq 2-\delta$, then

$$\Big\|\Pi_{s,\boldsymbol{t}}^n[f]\Big\|_0 \ll \mathrm{e}^{O(\delta n)}((\tau^{-\kappa/2}+\rho_1^{mn/8})\|f\|_0 + \rho_1^{mn\kappa/4}\tau^{-\kappa/2}\|f\|_1).$$

By choosing $n = c\log|\tau| + O(1)$ with $c = 4(m|\log\rho_1|)^{-1}$, and then τ_0 large enough and δ small enough in terms of κ, we may ensure that

$$\Big\|\Pi_{s,\boldsymbol{t}}^n[f]\Big\|_0 \leq \rho^n\|f\|_{1,\tau},$$

with $\rho = \rho_1^{m\kappa/10} < 1$ and as claimed. ☐

5.3.4. *Proof of Lemma 5.4.* Iterating the bound of Lemma 4.4, and using (4.6), we have for δ small enough and all $n \geq 0$ the bound

$$\Big\|\Pi_{s,\boldsymbol{t}}^n[f]\Big\|_1 \ll \mathrm{e}^{O(\delta n)}|\tau|^\kappa\|f\|_0 + \rho^n\|f\|_1,$$

for some $\rho \in [0,1)$. We replace f by $\Pi^n_{s,\boldsymbol{t}}[f]$ and use Lemmas 5.8 and 4.4. We find that for some constants $\tau_0 \geq 0$, $c_0 > 0$ and $\rho \in [0,1)$, if δ is small enough and $n = \lfloor c_0 \log \tau \rfloor$, then

$$\begin{aligned}\left\|\Pi^{2n}_{s,\boldsymbol{t}} f\right\|_{1,\tau} &\ll (\rho^n + \mathrm{e}^{O(\delta n)})\left\|\Pi^n_{s,\boldsymbol{t}} f\right\|_0 + |\tau|^{-\kappa}\rho^n\left\|\Pi^n_{s,\boldsymbol{t}} f\right\|_1 \\ &\ll (\rho^{2n} + \mathrm{e}^{O(\delta n)}\rho^n)\|f\|_0 + |\tau|^{-\kappa}(\rho^{2n} + \mathrm{e}^{O(\delta n)}\rho^n)\|f\|_1.\end{aligned}$$

At the cost of choosing c_0 large enough and δ small enough in terms of the implied constants, we obtain

$$\left\|\Pi^{2n}_{s,\boldsymbol{t}} f\right\|_{1,\tau} \leq \rho^{n/2}\|f\|_{1,\tau}.$$

By iterating, this bounds also holds for $n = k\lfloor c_0 \log \tau \rfloor$, $k \in \mathbb{N}$, from which we deduce by Gelfand's inequality that $\operatorname{srd}(\Pi_{s,\boldsymbol{t}}) \leq \rho^{1/4}$, and $\left\|(\mathrm{Id} - \Pi^{2n}_{s,\boldsymbol{t}})^{-1}\right\|_{1,\tau} \ll 1$. Finally, from the bounds

$$\left\|(\mathrm{Id} - \Pi_{s,\boldsymbol{t}})^{-1}\right\|_{1,\tau} \leq \left\|(\mathrm{Id} - \Pi^{2n}_{s,\boldsymbol{t}})^{-1}\right\|_{1,\tau} \sum_{0 \leq j < 2n} \left\|\Pi^j_{s,\boldsymbol{t}}\right\|_{1,\tau}$$

and $\left\|\Pi^j_{s,\boldsymbol{t}}\right\|_{1,\tau} \ll \mathrm{e}^{O(|\sigma-2|j)}$, we get the claimed result.

5.4. **Deduction of the meromorphic continuation.**

Proposition 5.9. *For some $\tau_0, t_0, \delta > 0$, and all $\|\boldsymbol{t}\| \leq t_0$, the function $s \mapsto \mathfrak{S}(s, \boldsymbol{t})$, initially only defined for* $\mathrm{Re}(s) > 2$*, has a meromorphic continuation to the set*

$$H := \left\{s \in \mathbb{C},\ s = \sigma + i\tau,\ \sigma \geq 2 - \delta\right\},$$

with possible poles occuring only for $|\tau| < \tau_0$ and $\lambda(s,t) = 1$. It is bounded uniformly in for $\mathrm{Re}(s) \geq 2 - \delta$ *and $|\tau| \geq \tau_0$ by*

$$|\mathfrak{S}(s, \boldsymbol{t})| \ll |\tau|^{O(\max(0, 2-\sigma))} \log(|\tau| + 2).$$

More precisely, for $|\tau| \leq \tau_0$, the function

$$s \mapsto \mathfrak{S}(s, \boldsymbol{t}) - \frac{\lambda(s, \boldsymbol{t})}{1 - \lambda(s, \boldsymbol{t})}(\Pi^{(0)}_{s,\boldsymbol{t}} + \Pi^{(1)}_{s,\boldsymbol{t}} + \cdots + \Pi^{(m-1)}_{s,\boldsymbol{t}})\mathbb{P}_{s,\boldsymbol{t}}[\mathbf{1}](1) \tag{5.7}$$

has an analytic continuation to $\sigma \geq 2 - \delta$ and $|\tau| \leq \tau_0$, and is uniformly bounded there.

Proof. We combine Lemmas 5.1, 5.2, 5.3 and 5.4 □

6. Asymptotic behaviour of the leading eigenvalue

In this section, we study the behaviour, for small t and s close to 2, of the leading eigenvalue $\lambda(s, \boldsymbol{t})$. The estimates in this section will reduce the problem to the estimation as $\boldsymbol{t} \to 0$ of the integral

$$\int_0^1 \mathrm{e}^{i\langle \boldsymbol{t}, \phi(x)\rangle} \xi(x)\,\mathrm{d}x,$$

where we recall the notation (2.2). We recall the hypotheses (2.3), (2.4).

6.1. **Perturbation theory and existence.** Let

$$\mathfrak{d} := -m \int_0^1 \log(x)\xi(x)\,\mathrm{d}x = \frac{m\pi^2}{12\log 2}. \tag{6.1}$$

Lemma 6.1. *For all small enough $\varepsilon > 0$, there exists $t_0 > 0$ such that whenever $|s-2| \leq \varepsilon$ and $\|\boldsymbol{t}\| \leq t_0$, we have*

$$\partial_{10}\lambda(s, \boldsymbol{t}) = -\mathfrak{d} + O(\varepsilon), \tag{6.2}$$

and

$$\lambda(s, \boldsymbol{t}) - 1 = \big(-\mathfrak{d} + O(\varepsilon_1)\big)(s-2) + O(\varepsilon_2). \tag{6.3}$$

Proof. Let $f_{s,\boldsymbol{t}} = \mathbb{P}_{s,\boldsymbol{t}}[\xi]$ denote an eigenfunction of $\Pi_{s,\boldsymbol{t}}$ associated with the eigenvalue $\lambda(s,\boldsymbol{t})$. By Lemma 4.3 and [Klo, Theorem 2.6], we have

$$\|f_{s,\boldsymbol{t}} - \xi\|_\infty \ll_\varepsilon |s-2| + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon}. \tag{6.4}$$

On the other hand, differentiating the eigenvalue equation and integrating with respect to the Lebesgue measure, we get

$$\partial_{10}\lambda(s,\boldsymbol{t})\int_{[0,1]} f_{s,\boldsymbol{t}}\,\mathrm{d}\nu = \int_{[0,1]}\Big((\log\mathfrak{T})g_{s,\boldsymbol{t}}f_{s,\boldsymbol{t}} + (g_{s,\boldsymbol{t}} - \lambda(s,\boldsymbol{t}))\partial_{10}f_{s,\boldsymbol{t}}\Big)\,\mathrm{d}\nu.$$

Here we recall the notation (4.2). Setting $(s,\boldsymbol{t}) = (2,\boldsymbol{0})$, with $f_{2,\boldsymbol{0}} = \xi$ and $g_{2,\boldsymbol{0}} = 1$, gives $\partial_{10}\lambda(2,\boldsymbol{0}) = -\mathfrak{d}$. Using the bound (6.4), we get the approximation (6.2). ☐

Lemma 6.2. *For all $\eta > 0$, there exists $t_0 > 0$ and a unique function $s_0 : [-t_0,t_0]^d \to \mathbb{C}$ such that $s_0(0) = 2$ and, for $\|\boldsymbol{t}\| \le t_0$,*

$$|s_0(\boldsymbol{t}) - 2| \le \eta, \qquad \lambda(s_0(\boldsymbol{t}),\boldsymbol{t}) = 1.$$

Proof. This follows from a general form of the implicit functions theorem, *e.g.* [Kum80, Theorem 1.1], whose hypotheses are satisfied by virtue of Lemma 6.1. ☐

In what follows we will not discuss the regularity of $s_0(\boldsymbol{t})$ at each $\boldsymbol{t}$: we are only interested about its asymptotic behaviour around $\boldsymbol{t} = 0$. We will use results on effective perturbation theory of linear operators, which have been worked out recently in [Klo].

6.2. **The sub-CLT case.**

Lemma 6.3. *For $\|\boldsymbol{t}\| \le t_0$, we have*

$$s_0(\boldsymbol{t}) - 2 = \frac{1}{\mathfrak{d}}\int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}(x)\rangle} - 1)\xi(x)\,\mathrm{d}x + O_\varepsilon(\|\boldsymbol{t}\|^2 + \|\boldsymbol{t}\|^{2\alpha_0-\varepsilon}).$$

Proof. By Theorem 2.6 of [Klo], we have

$$\begin{aligned}
\lambda(s,\boldsymbol{t}) &= \int_0^1 \Pi_{s,\boldsymbol{t}}\xi(x)\,\mathrm{d}x + O(\|\Pi_{s,\boldsymbol{t}} - \Pi_{2,0}\|^2)\\
&= (s-2)\int_0^1 \big[\tfrac{\partial}{\partial s}\Pi_{s,0}\xi\big]_{s=2}(x)\,\mathrm{d}x + \int_0^1 \Pi_{2,\boldsymbol{t}}\xi(x)\,\mathrm{d}x + O_\varepsilon(|s-2|^2 + \|\boldsymbol{t}\|^2 + \|\boldsymbol{t}\|^{2\alpha_0-\varepsilon})\\
&= (s-2)\int_0^1 \big[\tfrac{\partial}{\partial s}\,\mathbb{H}^m[g_{s,0}\xi]\big]_{s=2}(x)\,\mathrm{d}x + \int_0^1 \mathbb{H}^m[\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\rangle}\xi](x)\,\mathrm{d}x + O_\varepsilon(|s-2|^2 + \|\boldsymbol{t}\|^2 + \|\boldsymbol{t}\|^{2\alpha_0-\varepsilon})
\end{aligned}\tag{6.5}$$

by (4.2). Since $\int \mathbb{H}[f]\,\mathrm{d}\nu = \int f\,\mathrm{d}\nu$ and $\mathbb{H}[\xi] = \xi$, the first integral equals $m\int_0^1 \log(x)\xi(x)\,\mathrm{d}x = -\mathfrak{d}$. The second equals $\int_0^1 \mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}(x)\rangle}\xi(x)\,\mathrm{d}x$. For $\alpha = \min(\alpha_0,1)$, we have

$$\left|\int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\rangle} - 1)\xi\,\mathrm{d}\nu\right| \le \|\boldsymbol{t}\|^\alpha \int_0^1 \|\boldsymbol{\phi}\|^\alpha \xi\,\mathrm{d}\nu = \|\boldsymbol{t}\|^\alpha \int_0^1 \mathbb{H}[\|\boldsymbol{\phi}\|^\alpha\xi]\,\mathrm{d}\nu \ll \|\boldsymbol{t}\|^\alpha \tag{6.6}$$

by our hypothesis (2.3). Setting $s = s_0(\boldsymbol{t})$ and combining (6.5) and (6.6), we deduce $s_0(\boldsymbol{t}) - 2 = O(\|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0})$. Then another use of (6.5) yields our claimed estimate. ☐

6.3. **The CLT case.** Our next goal is to extract the term of order t^2 in the analysis above. We assume throughout that $\alpha_0 > 1$. In order to describe the order 2 coefficients, we introduce the following notation. Recalling (4.2), let

$$\mu_{\boldsymbol{\phi}} := \frac{1}{\mathfrak{d}}\int_0^1 \boldsymbol{\phi}(x)\xi(x)\,\mathrm{d}x,$$

$$\mathbb{K}[f] := \frac{1}{\xi}(\mathrm{Id} - \mathbb{H}^m)^{-1}\mathbb{N}^m[f\xi], \tag{6.7}$$

$$\boldsymbol{\psi} := \boldsymbol{\phi} + \mu_{\boldsymbol{\phi}}\log\mathfrak{T},$$

$$\boldsymbol{\chi} := \mathbb{K}\boldsymbol{\psi}. \tag{6.8}$$

The well-definedness of $\mu_{\boldsymbol{\phi}}$ follows from our hypothesis $\alpha_0 > 1$.

Lemma 6.4. *If $\alpha_0 > 1$, then*

$$s_0(\boldsymbol{t}) - 2 = \frac{1}{\mathfrak{d}} \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}(x)\rangle} - 1)\xi(x)\,\mathrm{d}x - \boldsymbol{t}^T C_{\boldsymbol{\phi}} \boldsymbol{t} + O_\varepsilon(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{\alpha_0+1-\varepsilon}), \tag{6.9}$$

with

$$C_{\boldsymbol{\phi}} = \frac{1}{\mathfrak{d}} \int_0^1 \Big(\frac{1}{2}(\boldsymbol{\psi} - \boldsymbol{\phi}) \cdot (\boldsymbol{\psi} - \boldsymbol{\phi})^T + \boldsymbol{\phi} \cdot (\boldsymbol{\psi} - \boldsymbol{\phi})^T + \boldsymbol{\psi}\mathbb{K}[\boldsymbol{\psi}]\Big)\xi\,\mathrm{d}\nu. \tag{6.10}$$

Moreover, if $\alpha_0 > 2$, then

$$s_0(\boldsymbol{t}) - 2 = i\langle \boldsymbol{t}, \mu_{\boldsymbol{\phi}}\rangle - \tfrac{1}{2}\boldsymbol{t}^T \Sigma_{\boldsymbol{\phi}} \boldsymbol{t} + O_\varepsilon(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon}), \tag{6.11}$$

with

$$\Sigma_{\boldsymbol{\phi}} := \frac{1}{\mathfrak{d}} \int_0^1 (\boldsymbol{\psi} + \boldsymbol{\chi} - \boldsymbol{\chi} \circ T^m) \cdot (\boldsymbol{\psi} + \boldsymbol{\chi} - \boldsymbol{\chi} \circ T^m)^T \xi\,\mathrm{d}\nu. \tag{6.12}$$

Remark. With the definition (6.12), it is clear that the matrix $\Sigma_{\boldsymbol{\phi}}$ is symmetric, positive semi-definite. It is definite if and only if the vectors $\{(\boldsymbol{\psi}+\boldsymbol{\chi}-\boldsymbol{\chi}\circ T^m)(x), x \in (0,1)\}$ span the whole space $\mathbb{R}^d$. The matrix $C_{\boldsymbol{\phi}}$ is well-defined whenever $\alpha_0 > 1$. The matrix $\Sigma_{\boldsymbol{\phi}}$ is well-defined whenever $\alpha_0 > 2$.

Proof. We extend the computations of Lemma 6.3, using our hypothesis on α_0 to expand the quantity $g_{s,\boldsymbol{t}} = \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle}\mathfrak{T}^{s-2}$ to order 2 at $s = 2$ and order 1 at $\boldsymbol{t} = 0$. Let

$$\Delta_{s,\boldsymbol{t}} = \Pi_{s,\boldsymbol{t}} - \Pi_{2,0}.$$

We write $\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle} = 1 + i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle + O((\|\boldsymbol{\phi}\|\|\boldsymbol{t}\|)^{\min(2,\alpha_0-\varepsilon)})$. Letting $s = s_0(\boldsymbol{t}) = 2 + O(\|\boldsymbol{t}\|)$, we obtain

$$\begin{aligned}\int_0^1 \Delta_{s,\boldsymbol{t}}\xi\,\mathrm{d}\nu &= \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle} - 1)\xi\,\mathrm{d}\nu - \mathfrak{d}(s-2) + i(s-2)\int_0^1 \langle \boldsymbol{t}, \boldsymbol{\phi}\rangle(\log\mathfrak{T})\xi\,\mathrm{d}\nu \\ &\quad + \frac{1}{2}(s-2)^2\int_0^1 (\log\mathfrak{T})^2\xi\,\mathrm{d}\nu + O(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{1+\alpha_0-\varepsilon}) \\ &= \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle} - 1)\xi\,\mathrm{d}\nu - \mathfrak{d}(s-2) - \boldsymbol{t}^T C_{1,\boldsymbol{\phi}}\boldsymbol{t} + O(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{1+\alpha_0-\varepsilon}),\end{aligned}$$

where $C_{1,\boldsymbol{\phi}} := \int_0^1 \boldsymbol{\phi} \cdot (\boldsymbol{\psi} - \boldsymbol{\phi})^T\xi\,\mathrm{d}\nu + \frac{1}{2}\int_0^1 (\boldsymbol{\psi} - \boldsymbol{\phi}) \cdot (\boldsymbol{\psi} - \boldsymbol{\phi})^T\xi\,\mathrm{d}\nu$. We use again Theorem 2.6 of [Klo], getting

$$\lambda(s, \boldsymbol{t}) = 1 + \int_0^1 \Delta_{s,\boldsymbol{t}}\xi\,\mathrm{d}\nu + \int_{[0,1]} \Delta_{s,\boldsymbol{t}}(\mathrm{Id} - \mathbb{H}^m)^{-1}(\mathrm{Id} - \mathbb{P})\Delta_{s,\boldsymbol{t}}\xi\,\mathrm{d}\nu + O(\|\Delta_{s,\boldsymbol{t}}\|^3).$$

By computations similar to (4.8), we have

$$\Big\|\Delta_{s,\boldsymbol{t}}[f] - i\,\mathbb{H}^m[\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle f] - i\mathfrak{d}^{-1}\Big(\int_0^1 \langle \boldsymbol{t}, \boldsymbol{\phi}\rangle\xi\,\mathrm{d}\nu\Big)\,\mathbb{H}^m[\log\mathfrak{T} f]\Big\|_0 \ll_\varepsilon (\|\boldsymbol{t}\|^{\alpha_0-\varepsilon} + \|\boldsymbol{t}\|^2)\|f\|_0.$$

Note that the left-hand side can be written $\|\Delta_{s,\boldsymbol{t}}[f] - i\langle \boldsymbol{t}, \mathbb{H}^m[\boldsymbol{\psi} f]\rangle\|_0$. We deduce

$$\begin{aligned}&\int_{[0,1]} \Delta_{s,\boldsymbol{t}}(\mathrm{Id} - \mathbb{H}^m)^{-1}(\mathrm{Id} - \mathbb{P})\Delta_{s,\boldsymbol{t}}\xi\,\mathrm{d}\nu \\ &\qquad = -\int_0^1 \langle \boldsymbol{t}, \boldsymbol{\psi}\rangle(\mathrm{Id} - \mathbb{H}^m)^{-1}(\mathrm{Id} - \mathbb{P})\,\mathbb{H}^m[\langle \boldsymbol{t}, \boldsymbol{\psi}\rangle\xi]\,\mathrm{d}\nu + O_\varepsilon(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{1+\alpha_0-\varepsilon}).\end{aligned}$$

Since $\mathbb{P}\,\mathbb{H}^m = \mathbb{P}$ and $\mathbb{H}^m = \mathbb{P} + \mathbb{N}^m$, we have $(\mathrm{Id} - \mathbb{H}^m)^{-1}(\mathrm{Id} - \mathbb{P})\,\mathbb{H}^m[f\xi] = \xi\mathbb{K}[f]$ where we recall the definition (6.7). Therefore, we have

$$-\int_0^1 \langle \boldsymbol{t}, \boldsymbol{\psi}\rangle(\mathrm{Id} - \mathbb{H}^m)^{-1}(\mathrm{Id} - \mathbb{P})\,\mathbb{H}^m[\langle \boldsymbol{t}, \boldsymbol{\psi}\rangle\xi]\,\mathrm{d}\nu = -\boldsymbol{t}^T C_{2,\boldsymbol{\phi}}\boldsymbol{t},$$

where $\mathbf{u}^T$ denotes the transpose of the column vector $\mathbf{u}$, and $C_{2,\boldsymbol{\phi}} = \int_{[0,1]} (\boldsymbol{\psi} \cdot \mathbb{K}[\boldsymbol{\psi}]^T)\xi\,\mathrm{d}\nu$. This proves (6.9) with $C_{\boldsymbol{\phi}} = C_{1,\boldsymbol{\phi}} + C_{2,\boldsymbol{\phi}}$ as claimed.

To prove (6.11), we note that by to the hypothesis $\alpha_0 > 2$, the quantity (6.12) is well-defined. In order to expand it, we first note that, with the definition (6.8), we have by construction $\mathbb{P}[\boldsymbol{\psi}\xi] = 0$, so that

$$\boldsymbol{\chi}\xi = (\mathrm{Id} - \mathbb{H}^m)^{-1}\mathbb{N}^m[\boldsymbol{\psi}\xi] = \sum_{j\geq 1}\mathbb{H}^{jm}[\boldsymbol{\psi}\xi] = \mathbb{H}^m[(\boldsymbol{\psi}+\boldsymbol{\chi})\xi].$$

By the property $\int f(g\circ T^m)\,\mathrm{d}\nu = \int \mathbb{H}^m[f]g\,\mathrm{d}\nu$, we have

$$\int(\boldsymbol{\psi}+\boldsymbol{\chi})\cdot(\boldsymbol{\chi}\circ T^m)^T\xi\,\mathrm{d}\nu = \int\mathbb{H}^m[(\boldsymbol{\psi}+\boldsymbol{\chi})\xi]\cdot\boldsymbol{\chi}^T\,\mathrm{d}\nu = \int\boldsymbol{\chi}\cdot\boldsymbol{\chi}^T\xi\,\mathrm{d}\nu.$$

Similarly, we have

$$\int(\boldsymbol{\chi}\circ T^m)\cdot(\boldsymbol{\chi}\circ T^m)^T\xi\,\mathrm{d}\nu = \int\mathbb{H}^m[(\boldsymbol{\chi}\circ T^m)\xi]\cdot\boldsymbol{\chi}^T\,\mathrm{d}\nu = \int\boldsymbol{\chi}\cdot\boldsymbol{\chi}^T\xi\,\mathrm{d}\nu.$$

We deduce that, with the definition (6.12), we have

$$\begin{aligned}\Sigma_{\boldsymbol{\phi}} &= \int(\boldsymbol{\psi}+\boldsymbol{\chi})\cdot(\boldsymbol{\psi}+\boldsymbol{\chi})^T\xi\,\mathrm{d}\nu - 2\int(\boldsymbol{\psi}+\boldsymbol{\chi})(\boldsymbol{\chi}\circ T^m)\,\mathrm{d}\nu - \int(\boldsymbol{\chi}\circ T^m)\cdot(\boldsymbol{\chi}\circ T^m)^T\xi\,\mathrm{d}\nu\\ &= \int\boldsymbol{\psi}\cdot\boldsymbol{\psi}^T\xi\,\mathrm{d}\nu + 2\int\boldsymbol{\psi}\cdot\mathbb{K}[\boldsymbol{\psi}]^T\xi\,\mathrm{d}\nu.\end{aligned}$$

On the other hand, expanding the squares in (6.10), we find

$$2C_{\boldsymbol{\phi}} = \int\boldsymbol{\psi}\cdot\boldsymbol{\psi}^T\xi\,\mathrm{d}\nu - \int\boldsymbol{\phi}\cdot\boldsymbol{\phi}^T\xi\,\mathrm{d}\nu + 2\int\boldsymbol{\psi}\cdot\mathbb{K}[\boldsymbol{\psi}]^T\xi\,\mathrm{d}\nu.$$

The claimed formula (6.11) follows by the Taylor expansion $\mathrm{e}^{iu} = 1 + u + \frac{1}{2}u^2 + O(u^{\alpha_0})$ with $u = i\langle\boldsymbol{t},\boldsymbol{\phi}\rangle$. □

7. Proof of Theorems 2.1, 1.1 and 1.2

Recall that Ω_Q consists in the rationals in $(0,1]$ of denominators at most Q, and let

$$\chi_Q(\boldsymbol{t}) := \sum_{x\in\Omega_Q}\exp(\langle i\boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle). \tag{7.1}$$

Proposition 7.1. *For all $\varepsilon > 0$, there exists $\delta, t_0 > 0$ such that for $\|\boldsymbol{t}\| \leq t_0$, we have*

$$\mathbb{E}_Q(\mathrm{e}^{i\langle\boldsymbol{t},S_{\boldsymbol{\phi}}(x)\rangle}) = Q^{s_0(\boldsymbol{t})-2}\{1 + O_\varepsilon(Q^{-\delta} + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\}.$$

Proof. Recall that $\chi_Q(\boldsymbol{t})$ was defined in (7.1), so that $\mathbb{E}_Q(\mathrm{e}^{i\langle\boldsymbol{t},S_{\boldsymbol{\phi}}(x)\rangle}) = \chi_Q(\boldsymbol{t})/\chi_Q(0)$. Let $\Omega\geq 1$ be a parameter, and $w:\mathbb{R}_+\to[0,1]$ be a smooth function satisfying

$$\mathbf{1}_{[0,1]} \leq w \leq \mathbf{1}_{[0,1+\Omega^{-1}]}, \qquad \|w^{(j)}\|_\infty \ll_j \Omega^j. \tag{7.2}$$

Then by a trivial bound on the contribution of $q\in[Q, Q(1+\Omega^{-1})]$, we have

$$\chi_Q(\boldsymbol{t}) = O(\Omega^{-1}Q^2) + \widetilde{\chi}_Q(\boldsymbol{t}), \qquad \text{where} \quad \widetilde{\chi}_Q(\boldsymbol{t}) := \sum_{\substack{1\leq a\leq q\\(a,q)=1}}\mathrm{e}^{i\langle\boldsymbol{t},S_{\boldsymbol{\phi}}(a/q)\rangle}w\Big(\frac{q}{Q}\Big). \tag{7.3}$$

By Perron's formula, we have

$$\widetilde{\chi}_Q(\boldsymbol{t}) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)\,\mathrm{d}s,$$

where the Mellin transform $\widehat{w}(s) = \int_0^\infty w(u)u^{s-1}\,\mathrm{d}u$ is defined for $\mathrm{Re}(s) > 0$. We move the contour to the line $\mathrm{Re}(s) = 2-\delta$. If $t_0,\delta > 0$ are small enough, then by Proposition 5.9, we encounter exactly one pole, at $s = s_0(\boldsymbol{t})$. By Cauchy's theorem, we deduce

$$\widetilde{\chi}_Q(\boldsymbol{t}) = \operatorname*{Res}_{s=s_0(\boldsymbol{t})}(Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)) + \frac{1}{2\pi i}\int_{2-\delta-i\infty}^{2-\delta+i\infty}Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)\,\mathrm{d}s. \tag{7.4}$$

For some absolute constant $C\geq 1$, Proposition 5.9 yields the bound

$$|\mathfrak{S}(s,\boldsymbol{t})| \ll (|\tau|+1)^C.$$

On the other hand, we have $|\widehat{w}(s)| \ll \Omega^{C+2}|s|^{-C-2}$ for $\mathrm{Re}(s) \in [1/2, 3]$ by integration by parts and (7.2), so that

$$\left|\int_{2-\delta-i\infty}^{2-\delta+i\infty} Q^s \mathfrak{S}(s, \boldsymbol{t})\widehat{w}(s)\,\mathrm{d}s\right| \ll \Omega^{C+2}Q^{2-\delta}. \tag{7.5}$$

Finally, we have by (5.7)

$$\operatorname*{Res}_{s=s_0(\boldsymbol{t})}(Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)) = -\frac{Q^{s_0}\widehat{w}(s_0)}{\partial_{10}\lambda(s_0,\boldsymbol{t})}\sum_{0\le j<m}\Pi^{(j)}_{s_0,\boldsymbol{t}}\mathbb{P}_{s_0,\boldsymbol{t}}[\mathbf{1}](1), \tag{7.6}$$

where we abbreviated $s_0 = s_0(\boldsymbol{t})$ in the right-hand side. By perturbation theory and Lemma 4.3, we have

$$\Pi^{(j)}_{s_0,\boldsymbol{t}}\mathbb{P}_{s_0,\boldsymbol{t}}[\mathbf{1}](1) = \frac{1}{2\log 2}\{1 + O_\varepsilon(\|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\}$$

for $0 \le j < m$. The quantity $\partial_{10}\lambda(s_0(\boldsymbol{t}),\boldsymbol{t})$ was estimated in (6.2). Finally, we have by (7.2)

$$\widehat{w}(s_0) = \frac{1}{2}\{1 + O_\varepsilon(\Omega^{-1} + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\}.$$

Inserting these estimates in (7.6), we deduce

$$\operatorname*{Res}_{s=s_0(\boldsymbol{t})}(Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)) = \frac{3}{\pi^2}Q^{s_0(\boldsymbol{t})}\{1 + O_\varepsilon(\Omega^{-1} + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\}.$$

Grouping this (7.3)(7.4) and (7.5), we conclude

$$\chi_Q(\boldsymbol{t}) = \frac{3}{\pi^2}Q^{s_0(\boldsymbol{t})}\{1 + O_\varepsilon(\Omega^{-1} + \Omega^{C+2}Q^{-\delta} + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\}.$$

Our claim follows by optimizing $\Omega = Q^{\delta/(C+3)}$ and dividing by $\chi_Q(0)$. □

Proof of Theorem 2.1. We use Proposition 7.1 along with Lemmas 6.3–6.4 and the value (6.1). □

Proof of Theorem 1.1. We let $m = 1$ in Theorem 2.1. □

8. Applications

When we will use the Berry–Esseen inequality, we will require a separate treatment of very small values of t in order to handle the error term $O(Q^{-\delta})$ in Theorem 2.1 (the argument described in [Gou15, Remark 3.8] is not readily adapted since part of this term originates from counting pairwise coprime numbers).

Lemma 8.1. *Suppose that the function* $\boldsymbol{\phi}$ *satisfies* (2.3) *and* (2.4)*. Then we have*

$$\mathbb{E}_Q(\mathrm{e}^{i\langle \boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle}) = 1 + O(\|\boldsymbol{t}\|^{\alpha_0/3}\log Q). \tag{8.1}$$

Proof. For all $n \ge 1$, define $c(n) := \sup_{x\in(\frac{1}{n+1},\frac{1}{n})}\|\boldsymbol{\phi}(x)\|^{\alpha_0/3}$. Then with the terminology of [BH08, p.750], the function c has strong moments to order 3, so that $\mathbb{E}_Q(\sum_{j=1}^r c(a_j)) \ll \log Q$ by [BH08, Remark 1.2]. We conclude by taking expectations in the bound $\left|\mathrm{e}^{i\langle \boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle} - 1\right| = O(\sum_{j=1}^r(\|\boldsymbol{t}\|\|\boldsymbol{\phi}(T^{j-1}(x))\|)^{\alpha_0/3})$. □

Proof of Theorem 1.2. In (1.6), we estimate the integral in $\mathfrak{I}_\phi(t)$ by Lemma A.1. From (6.12), and with the notation (1.7), we obtain

$$U(t) = it\mu - \frac{t^2}{2}\sigma + O_\varepsilon(|t|^{\min(3,\alpha_0-\varepsilon)}), \tag{8.2}$$

where, with $\psi(x) = \phi(x) + \mu\log x$,

$$\sigma = \frac{12\log 2}{\pi^2}\int_0^1(\psi(x) + \chi(x) - \chi(T(x)))^2\frac{\mathrm{d}x}{1+x}.$$

We recall that χ is related to ψ by (6.8) (with $m = 1$). It is clear that $\sigma \geq 0$. If $\sigma = 0$, then the integrand vanishes identically, and we would conclude that $\phi = -\mu \log -\chi + \chi \circ T$, contradicting our hypothesis. The bounds (8.2), (8.1) and (1.5) may now be used in conjuction with the Berry-Esseen theorem [Ten15, theorem II.7.16] to conclude. $\square$

8.1. **Large moments of continued fractions expansions.** Let $\lambda \geq 1/2$, and M_λ be defined in (1.9). The function $\phi_\lambda(x) := \lfloor 1/x \rfloor^\lambda$ and satisfies the hypotheses of Theorem 1.1 with $\alpha_0 = 1/\lambda - \varepsilon$, $\kappa_0 = 1$, and all small enough exponents $\lambda_0 > 0$.

8.1.1. *Case* $\lambda = 1/2$. When $\lambda = 1/2$, the estimate (1.6) holds, and the integral is evaluated by means of Lemma A.8. With the notation $\sigma = (\pi^2/6)^{-1/2}$ and

$$\chi_{1/2,Q}(t) := \mathbb{E}_Q\Big(\exp\Big\{ it \frac{M_{1/2}(x) - \mu \log Q}{\sigma\sqrt{\log Q \log\log Q}} \Big\} \Big),$$

we find that for $|t| \leq \log\log Q$,

$$\chi_{1/2,Q}(t) = \exp\Big\{ -\frac{3t^2}{\pi^2} + O\Big(\frac{1}{Q^\delta} + \frac{|t|}{(\log Q)^{1/2-\varepsilon}} + \frac{t^2}{(\log\log Q)^{1-\varepsilon}} \Big) \Big\}.$$

On the other hand, by (8.1), we have $\chi_{1/2,Q}(t) = 1 + O(|t|^{1/2} \log Q)$. Inserting these two bounds in the Berry-Esseen theorem [Ten15, theorem II.7.16] yields the claimed conclusion (1.11).

8.1.2. *Case* $\lambda = 1$. When $\lambda = 1$, we use the estimate (1.5). Define

$$\chi_{1,Q}(t) := \mathbb{E}_Q\Big(\exp\Big\{ it\Big(\frac{M_1(x)}{\log Q} - \frac{\log\log Q - \gamma_0}{\pi^2/12} \Big) \Big\} \Big).$$

Then for $0 < t \leq (\log Q)^{1-\varepsilon}$, we obtain, using Lemma A.7,

$$\chi_{1,Q}(t) = \exp\Big\{ -\frac{it}{\pi^2/12}\Big(|\log t| - \pi i \Big) + O\Big(\frac{1}{Q^\delta} + \frac{|t|^{1-\varepsilon} + |t|^{2-\varepsilon}}{(\log Q)^{1-\varepsilon}} \Big) \Big\},$$

and we may again conclude by the Berry-Esseen inequality.

8.1.3. *Case* $\lambda \notin \{1/2, 1\}$. Assume first $\lambda > 1$. Then we use the estimate (1.5). Define

$$\chi_{\lambda,Q}(t) := \mathbb{E}_Q\Big(\exp\Big\{ it \frac{M_\lambda(x)}{(\log Q)^\lambda} \Big\} \Big).$$

Then for $0 \leq t \leq \log\log Q$, we obtain

$$\chi_{\lambda,Q}(t) = \exp\Big\{ \frac{c_* \log 2}{\pi^2/12} t^{1/\lambda} + O\Big(\frac{1}{Q^\delta} + \frac{t^{1/\lambda - \varepsilon}}{(\log Q)^{1-\varepsilon}} + \frac{t^{1/\lambda}}{(\log\log Q)^{1-\varepsilon}} \Big) \Big\},$$

where c_* is given in Lemma A.8. In particular, with the definition of $c_{1/\lambda}$ given in (1.10), we find

$$\frac{c_* \log 2}{\pi^2/12} = -(c_{1/\lambda})^{1/\lambda}(1 - i \tan(\tfrac{\pi}{2\lambda})),$$

and we may again conclude by the Berry-Esseen inequality and Lemma 8.1.

The case $\lambda \in (1/2, 1)$ follows by identical computations, the shift by $\mu \log Q$ being accounted for by the term $c_1 t$ in Lemma A.8.

8.2. **Central values of L-functions and modular symbols.** Let $f(z) = \sum_{n\geq 1} a_n \mathrm{e}(nz)$ be a non-zero primitive Hecke eigencuspform of weight k for $SL(2,\mathbb{Z})$ with trivial multiplier. Note that k is necessarily even and $k \geq 12$.

Define, for all integer $1 \leq m \leq k-1$ and all $x \in \mathbb{Q}$, the modular symbol

$$\langle x\rangle_{f,m} := \frac{(2\pi i)^m}{(m-1)!}\int_x^{i\infty} f(z)(z-x)^{m-1}\,\mathrm{d}z. \tag{8.3}$$

Lemma 8.2. *For $m > k/2$, the function $x \mapsto \langle x\rangle_{f,m}$, initially defined over $\mathbb{Q}$, can be extended to a bounded function in $H^{1-\varepsilon}(\mathbb{R})$ for any $\varepsilon \in (0,1)$.*

Proof. By Deligne's bound [Del74], we have $|a_n| \ll_{\varepsilon,f} n^{(k-1)/2+\varepsilon}$. Therefore $\sum_{n\geq 1} |a_n|/n^m < \infty$, and we deduce by Fubini's theorem that for $x \in \mathbb{Q}$,

$$\langle x\rangle_{f,m} = (-1)^m \sum_{n\geq 1} \frac{a_n \mathrm{e}(nx)}{n^m},$$

and the left-hand side is now defined for $x \in \mathbb{R}$. By [Iwa97, Theorem 5.3], we have

$$\sum_{n\leq t} a_n \mathrm{e}(nx) \ll_f t^{k/2}\log(2t) \qquad (t \geq 1) \tag{8.4}$$

uniformly in $x \in \mathbb{R}$. Let $x, x' \in \mathbb{R}$, $\delta = |x - x'|$, and for $t \geq 1$, $S(t) := \sum_{n\leq t} a_n(\mathrm{e}(nx) - \mathrm{e}(nx'))$. Then using (8.4) and partial summation, we obtain $|S(t)| \ll_{\varepsilon,f} t^{k/2+\varepsilon}\min(1,\delta t)$, and so

$$\begin{aligned}|\langle x\rangle_{f,m} - \langle x'\rangle_{f,m}| &\ll_{\varepsilon,f} \int_1^\infty t^{-m-1+k/2+\varepsilon}\min(1,\delta t)\,\mathrm{d}t\\ &\ll_{\varepsilon,f} \delta + \delta^{m-k/2-\varepsilon}\end{aligned}$$

as claimed, since $m \geq k/2+1$. ☐

Lemma 8.3. *For any $\varepsilon \in (0,1)$, some function $\phi_f \in H^{1-\varepsilon}([0,1],\mathbb{C})$, and all $x \in \mathbb{Q}\smallsetminus\{0\}$, we have*

$$\langle x\rangle_{f,k/2} = (-i)^{k/2-1}\langle \tfrac{-1}{x}\rangle_{f,k/2} + \phi_f(x).$$

Moreover, we have uniformly $\|\phi_f \circ h\|_{(1-\varepsilon)} \ll 1$ uniformly for $h \in \mathcal{H}^4$.

Proof. For $\mathrm{Im}(z) > 0$, define

$$\begin{aligned}\widehat{f}(z) &:= \sum_{n\geq 1} \frac{a_n}{n^{k-1}}\mathrm{e}(nz)\\ &= \sum_{n\geq 1} a_n \mathrm{e}(nz)\frac{(-i)^{k-1}}{(k-1)!}\int_0^{i\infty} \tau^{k-2}\mathrm{e}^{2\pi i n\tau}\,\mathrm{d}\tau\\ &= \frac{(-i)^{k-1}}{(k-1)!}\int_z^{i\infty} (\tau - z)^{k-2} f(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Using the modularity relation $f(-1/z) = z^k f(z)$, and making the change of variables $\tau \to -1/\tau$, we obtain

$$\begin{aligned}\frac{(k-1)!}{(-i)^{k-1}} z^{k-2}\widehat{f}(-1/z) &= z^{k-2}\int_{-1/z}^{i\infty} (\tau + 1/z)^{k-2} f(\tau)\,\mathrm{d}\tau = \int_z^0 (-1/\tau + 1/z)^{k-2}(z\tau)^{k-2} f(\tau)\,\mathrm{d}\tau\\ &= -\int_0^z (\tau - z)^{k-2} f(\tau)\,\mathrm{d}\tau\end{aligned}$$

and so the quantity

$$r_f(z) := \widehat{f}(z) - z^{k-2}\widehat{f}(-1/z) = \frac{(-i)^{k-1}}{(k-1)!}\int_0^{i\infty} (\tau - z)^{k-2} f(\tau)\,\mathrm{d}\tau \qquad (\mathrm{Im}(z) > 0)$$

is a polynomial in z of degree $k-2$.

Let now $x \in \mathbb{Q}_{>0}$. As $\delta \to 0$ with $\delta > 0$, we have

$$\begin{aligned}\widehat{f}(x(1+i\delta)) &= \frac{(-i)^{k-1}}{(k-1)!}\int_{x(1+i\delta)}^{i\infty}(\tau - x - ix\delta)^{k-2}f(\tau)\,\mathrm{d}\tau \\ &= \frac{(-i)^{k-1}}{(k-1)!}\Big(\int_x^{i\infty} - \int_x^{x(1+i\delta)}\Big)(\tau - x - ix\delta)^{k-2}f(\tau)\,\mathrm{d}\tau.\end{aligned}$$

The second integral is $O(\delta^M)$ as $\delta \to 0$ for any fixed $M > 0$, since f is a cusp form. Thus, by the binomial formula, we obtain

$$\widehat{f}(x(1+i\delta)) = \frac{(-1)^{k-1}}{(k-1)(2\pi)^{k-1}}\sum_{\ell=0}^{k-2}\frac{(2\pi x\delta)^\ell}{\ell!}\langle x\rangle_{f,k-1-\ell} + o(\delta^{k-2}).$$

In the same way, since $-\frac{1}{x(1+i\delta)} = -\frac{1}{x}(1-i\delta')$ with $\delta' = \delta/(1+i\delta)$, so that $\mathrm{Re}(\delta) > 0$, we have

$$\begin{aligned}&(x(1+i\delta))^{k-2}\widehat{f}(-\frac{1}{x(1+i\delta)}) \\ &= \frac{(-1)^{k-1}}{(k-1)(2\pi)^{k-1}}\sum_{\ell=0}^{k-2}(i\delta)^\ell\sum_{j=0}^{\ell}\frac{(-1)^j}{j!}\binom{k-2-j}{\ell-j}(2\pi)^j x^{k-2-j}\left\langle\frac{-1}{x}\right\rangle_{f,k-1-j} + o(\delta^{k-2}).\end{aligned}$$

With $m := k/2-1$, reading the coefficients of δ^m on each side of the definition of $r_f(x(1+i\delta))$, and since k is even, we deduce

$$\langle x\rangle_{f,1+m} - (-i)^m\sum_{j=0}^{m}c_{j,k}x^j\left\langle\frac{-1}{x}\right\rangle_{f,1+m+j} = -(k-1)(2\pi)^{m+1}i^m r_f^{(m)}(x),$$

$$c_{j,k} := j!\binom{m}{j}\binom{m+j}{j}\Big(\frac{-1}{2\pi}\Big)^j.$$

We single out the term $j = 0$. The function

$$\phi_f(x) := (-i)^m\sum_{j=1}^{m}c_{j,k}x^j\left\langle\frac{-1}{x}\right\rangle_{f,1+m+j} - (k-1)(2\pi)^{m+1}i^m r_f^{(m)}(x),$$

defines, by Lemma 8.2, a function in $H^{1-\varepsilon}([\frac{1}{n+1},\frac{1}{n}])$ for all $\varepsilon \in (0,1)$ and $n \geq 1$. The value $c_{0,k} = 1$ proves our claimed formula. Finally, for $h \in \mathcal{H}$, by the rules (3.1), (3.2), (3.4) and 1-periodicity of $x \mapsto \langle x\rangle_{f,1+m+j}$, we have

$$\begin{aligned}\|\phi_f\circ h\|_{(1-\varepsilon)} &\ll_f \sum_{j=1}^{m}\left\|x\mapsto h(x)^j\langle x\rangle_{f,1+m+j}\right\|_{(1-\varepsilon)} + \left\|r_f^{(m)}\circ h\right\|_{(1-\varepsilon)} \\ &\ll_f \sum_{j=1}^{m}(\left\|(h^j)'\right\|_\infty^\varepsilon\left\|h^j\right\|_\infty^{1-\varepsilon} + \left\|h^j\right\|_\infty) + \|h'\|_\infty^{1-\varepsilon} \\ &\ll_f 1.\end{aligned}$$

By the rule (3.2) again, we deduce that the same bound $\|\phi_f\circ h\|_{(1-\varepsilon)} \ll 1$ holds for $h \in \mathcal{H}^4$. $\square$

Proof of Corollary 1.5. Let $\omega := (-i)^{k/2-1}$, so that $\omega^4 = 1$. Iterating Lemma 8.3, we have for all $x \in \mathbb{Q}\cap(0,1]$,

$$\langle x\rangle_{f,k/2} = \sum_{j=1}^{r}\omega^{j-1}\phi((-1)^{j-1}T^{j-1}(x)) + \langle 0\rangle_{f,k/2}. \tag{8.5}$$

Note that changing the coordinates of $\mathscr{R}$ by an amount $O(1/\sqrt{\log Q})$ in (1.17) does not change the right-hand side, so that we may replace $\langle x\rangle_{f,k/2}$ by $\langle x\rangle_{f,k/2} - \langle 0\rangle_{f,k/2}$. For all $\boldsymbol{t} \in \mathbb{R}^2$, identifying $\mathbb{C} \simeq \mathbb{R}^2$ with basis $(1, i)$, we let

$$\chi(t) := \mathbb{E}_Q\Big(\exp\Big\{ \frac{i\langle \boldsymbol{t}, \langle x\rangle_{f,k/2} - \langle 0\rangle_{f,k/2}\rangle}{\sigma_f \sqrt{\log Q}} \Big\}\Big),$$

The j-th summand in (8.5) only depends on $j \pmod 4$. We apply Theorem 2.1 with $m = 4$ and $d = 2$. The hypothesis (2.3) is satisfied with $\alpha_0 = 4$. The hypothesis (2.4) is satisfied for any $\lambda_0 < \frac{1}{2-2\varepsilon}$, by using Lemma 8.3 and noting that $\left\|\phi_f|_{h(\mathcal{I})}\right\|_{(1-\varepsilon)} \ll |h'(0)|^{-1+\varepsilon} \|\phi_f \circ h\|_{(1-\varepsilon)}$. We deduce using (2.6) along with the expressions (6.11), (6.12), we obtain for some $\mu_f \in \mathbb{R}^2$ and real 2×2 matrix Σ_f the estimate

$$\chi(\boldsymbol{t}) = \exp\Big\{ i\langle \boldsymbol{t}, \mu_f\rangle - \tfrac{1}{2}\boldsymbol{t}^T \Sigma_f \boldsymbol{t} + O\Big(\frac{\|\boldsymbol{t}\| + \|\boldsymbol{t}\|^3}{\sqrt{\log Q}} + \frac{1}{Q^\delta} \Big)\Big\}.$$

To compute the variance, we appeal to the bound

$$\mathbb{E}_Q(|\langle x\rangle_{f,m} - \mu_f \log Q|^4) \ll (\log Q)^2. \tag{8.6}$$

This can be proved by shifting to the setting of [BV05a], where the variable t is extended to a complex neighborhood of the origin; the functions U, V defined in (2.5) are, in this case, analytic in t near the origin, by boundedness of $\boldsymbol{\phi}$. Then by *e.g.* [Bil95, th. 25.12] and (8.6), we find

$$\mu_f = \lim_{Q\to\infty} \frac{\mathbb{E}_Q(\langle x\rangle_{f,k/2})}{\log Q}, \qquad \Sigma_f = \lim_{Q\to\infty} \frac{\mathbb{E}_Q(\langle x\rangle_{f,k/2} \langle x\rangle_{f,k/2}^T)}{\sigma_f^2 \log Q}.$$

On the other hand, as $Q \to \infty$, we have the following asymptotic formulae, where now $\langle x\rangle_{f,k/2}$ is interpreted as a complex number:

$$\mathbb{E}_Q(\langle x\rangle_{f,m}) = o(\log Q), \tag{8.7}$$

$$\mathbb{E}_Q(\langle x\rangle_{f,m}^2) = o(\log Q), \tag{8.8}$$

$$\mathbb{E}_Q(|\langle x\rangle_{f,m}|^2) \sim 2\sigma_f^2 \log Q. \tag{8.9}$$

These statements can be proven (in a stronger form) by standard methods, using orthogonality of additive characters, the approximate functional equation [IK04, Theorem 5.3] and Rankin-Selberg theory [Iwa97, Chapter 13.6]. The value σ_f appears as $\sigma_f^2 = \mathrm{Res}_{s=1} L(f \times \overline{f}, 1)$, which is evaluated in [Iwa97, eq. (13.52)]. Note that the analogues of (8.8) and (8.9) with a single average over numerator have recently been computed in [BFK+]; in their result as stated, however, the denominator is assumed to be prime.

The equality (8.7) shows that $\mu_f = 0$. The equality (8.8) shows that the matrix Σ_f is a multiple of the identity, and the equality (8.9) then shows that $\Sigma_f = \mathrm{Id}$. Using the Berry–Esseen inequality, along with Lemma 8.1, concludes the proof of Corollary 1.5.

□

8.3. **Sums of quadratic functions.** Let $D \in \mathbb{N}_{>0}$ be given, whith is not a perfect square, and

$$Q_D = \{Q(X) = aX^2 + bX + c, (a,b,c) \in \mathbb{Z}^3, b^2 - 4ac = D\}.$$

If $Q \in Q_D$, we denote a_Q, b_Q, c_Q its coefficients. For any given $x \in \mathbb{Q}$, by [Zag99, p. 1147], there are only finitely many $Q \in Q_D$ such that $a_Q < 0$ and $Q(x) > 0$, so that the for all even $k > 2$, the expression

$$F_k(x) := \sum_{\substack{Q\in Q_D \\ a_Q<0 \\ Q(x)>0}} (Q^{k-1})^{(k-1)}(x)$$

is well-defined.

As in [Zag75, KZ84], we introduce the weight $2k$ holomorphic cusp form

$$f_{D,k}(z) := \frac{D^{k-1/2}}{2\pi\binom{2k-2}{k-1}} \sum_{Q\in\mathcal{Q}_D} \frac{1}{Q(z)^k}. \tag{8.10}$$

Lemma 8.4. *For all even $k \geq 2$, we have*

$$F_k(x) = \frac{(2k-2)!}{(2\pi)^k(-1)^{k/2}} \operatorname{Im}(\langle x\rangle_{f_{D,k},k}).$$

Note that for $k \in \{2,4\}$, both sides are zero.

Proof. For all $Q \in \mathcal{Q}_D$ and $x \in \mathbb{Q}$, we compute

$$R_Q(x) := \operatorname{Im} \int_x^{i\infty} \frac{(z-x)^{k-1}}{Q(z)^k} \,\mathrm{d}z.$$

This is trivially invariant by $Q \leftarrow (-Q)$, so we may assume $a_Q > 0$. By Cauchy's formula, when we shift the contour horizontally, the contribution from the real segment is a real number, so does not contribute, unless we pass a pole. As we send to $+\infty$, or $-\infty$, the integral along the vertical half-line vanishes. Therefore $R_Q(x) = 0$ if $Q(x) > 0$. If $Q(x) < 0$, we send the integral to $+\infty$, and pick up a half a residue, coming from the largest root of Q. It is convenient to set

$$x = \frac{-b+\sqrt{D}}{2a} - \frac{\sqrt{D}}{a}t, \qquad Q(x) = -\frac{D}{a}t(1-t), \qquad ((a,b) = (a_Q, b_Q))$$

so that

$$(Q^{k-1})^{(k-1)}(x) = D^{(k-1)/2} \frac{\mathrm{d}^{k-1}}{\mathrm{d}y^{k-1}}(y(1-y))^{k-1}\Big|_{y=t}.$$

With $\alpha = \frac{-b-\sqrt{D}}{2a}$ and $\beta = \frac{-b+\sqrt{D}}{2a}$, we have then

$$\begin{aligned} R_Q(x) &= \operatorname{Im}\Big(-\pi i \operatorname{Res}_{z=\beta} \frac{(z-x)^{k-1}}{a^k(z-\alpha)^k(z-\beta)^k}\Big) \\ &= -\frac{\pi}{D^{k/2}} \sum_{j=0}^{k-1} \binom{k-1}{j}\binom{-k}{j} t^j \\ &= -\frac{\pi}{(k-1)!D^{k/2}} \frac{\mathrm{d}^{k-1}}{\mathrm{d}y^{k-1}}(y(1-y))^{k-1}\Big|_{y=t}, \end{aligned}$$

so that for $Q(x) < 0$, and $a_Q > 0$,

$$R_Q(x) = \frac{-\pi}{(k-1)!D^{k-1/2}} (Q^{k-1})^{(k-1)}(x).$$

We deduce from the above, and changing Q to $-Q$, that

$$\begin{aligned} F_k(x) &= -\sum_{\substack{Q\in\mathcal{Q}_D \\ a_Q>0 \\ Q(x)<0}} (Q^{k-1})^{(k-1)}(x) \\ &= \frac{(k-1)!D^{k-1/2}}{\pi} \sum_{\substack{Q\in\mathcal{Q}_D \\ a_Q>0}} R_Q(x) \\ &= \frac{(k-1)!D^{k-1/2}}{2\pi} \sum_{Q\in\mathcal{Q}_D} R_Q(x) \end{aligned}$$

since, as k is even, changing Q to $-Q$ does not alter $R_Q(x)$. Therefore, adapting the computations of pages 223-226 of [KZ84], we have

$$\begin{aligned}
\operatorname{Im}(\langle x\rangle_{f_{D,k},k}) &= \frac{(2\pi i)^k}{(k-1)!}\operatorname{Im}\int_x^{i\infty}(z-x)^{k-1}f_{D,k}(z)\,\mathrm{d}z \\
&= c_1\lim_{\varepsilon\to 0}\operatorname{Im}\int_{x+i\varepsilon}^{x+i/\varepsilon}(z-x)^{k-1}\sum_{Q\in Q_D}\frac{1}{Q(z)^k}\,\mathrm{d}z \\
&= c_1\lim_{\varepsilon\to 0}\sum_{Q\in Q_D}\operatorname{Im}\int_{x+i\varepsilon}^{x+i/\varepsilon}(z-x)^{k-1}\frac{1}{Q(z)^k}\,\mathrm{d}z \\
&= c_2F_k(x) - c_1\operatorname{Im}\lim_{\varepsilon\to 0}(A_\varepsilon+B_\varepsilon), \qquad (8.11)
\end{aligned}$$

where $c_1=(2\pi i)^kD^{k-1/2}/(2\pi(k-1)!\binom{2k-2}{k-1})$, $c_2=(2\pi i)^k/(2k-2)!$, and

$$A_\varepsilon=\sum_{Q\in Q_D}\int_x^{x+i\varepsilon}\frac{(z-x)^{k-1}}{Q(z)^k}\,\mathrm{d}z,\qquad B_\varepsilon=\sum_{Q\in Q_D}\int_{x+i/\varepsilon}^{i\infty}\frac{(z-x)^{k-1}}{Q(z)^k}\,\mathrm{d}z.$$

As in [KZ84, p.226], we factor from the sum over Q the orbit under the stabilizer Γ_x of x in $SL_2(\mathbb{Z})$, which is a cyclic group. Let σ denote a generator of Γ_x. Then

$$\begin{aligned}
A_\varepsilon &= \sum_{Q\in Q_D}\int_0^{i\varepsilon}\frac{z^{k-1}\,\mathrm{d}z}{Q(x+z)^k} \\
&= \sum_{Q\in Q_D/\Gamma_x}\sum_{n\in\mathbb{Z}}\int_0^{i\varepsilon}\frac{z^{k-1}\,\mathrm{d}z}{(\sigma^n\cdot Q)(x+z)^k}.
\end{aligned}$$

Here, for $\sigma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, we let $(\sigma\cdot Q)(X)=(cX+d)^2Q(\frac{aX+b}{cX+d})$. For $x=p/q$ in reduced form, we pick $\sigma=\left(\begin{smallmatrix}1-pq&p^2\\-q^2&1+pq\end{smallmatrix}\right)$, so that

$$(\sigma^n\cdot Q)(x+z)=(1-nq^2z)^2Q\Big(x+\frac{z}{1-nq^2z}\Big),\qquad (x=p/q).$$

Then

$$A_\varepsilon=\sum_{Q\in Q_D/\Gamma_x}\sum_{n\in\mathbb{Z}}\int_0^{i\varepsilon}\frac{z^{k-1}\,\mathrm{d}z}{[(1-nzq^2)^2Q(x+z/(1-nzq^2))]^k}.$$

Changing n into $n'=n+\delta_{Q,x}$ with $\delta_{Q,x}:=Q'(x)/(2q^2Q(x))$, where n' runs over $\mathbb{Z}+\delta_{Q,x}$, we have

$$\begin{aligned}
\Big|(1-nzq^2)^2Q(x+z/(1-nzq^2))\Big| &= \Big|Q(x)\Big((1-n'q^2z)^2+Dz^2/(4Q(x)^2)\Big)\Big| \\
&\gg_x |Q(x)|(1+|n'z|)^2
\end{aligned}$$

since $|Q(x)|\geq 1/q^2$, and $z\in i\mathbb{R}$. Then

$$\begin{aligned}
A_\varepsilon &\ll_{D,k}\varepsilon^k\sum_{Q\in Q_D/\Gamma_x}|Q(x)|^{-k}\sum_{n'\in\mathbb{Z}+\delta_{Q,x}}\int_0^1\frac{t^{k-1}\,\mathrm{d}t}{(1+|n'\varepsilon t|)^{2k}} \\
&\ll_{x,D,k}\varepsilon^{k-1}.
\end{aligned}$$

For the term B_ε, we change variables $z\to x-1/z$, and then $Q(X)\to X^2Q(-1/X)$, getting

$$B_\varepsilon=\sum_{Q\in Q_D}\int_0^{i\varepsilon}\frac{z^{k-1}\,\mathrm{d}z}{[(1-xz)^2Q(z/(1-xz))^k},$$

and from here the argument for A_ε can be reproduced, by factoring the sum over the stabilizer Γ_0 of the cusp 0. We find again $B_\varepsilon\ll_{x,D,k}\varepsilon^{k-1}$, and so for each fixed $x\in\mathbb{Q}$, $\lim_{\varepsilon\to 0}(A_\varepsilon+B_\varepsilon)=0$. Using (8.11) and the definition (8.3), the Lemma follows. ☐

Proof of Corollary 1.6. Combine Lemma 8.4, and Corollary 1.5 with $f = f_{D,k}$ and $\mathscr{R} = \mathbb{R} \times (-\infty, v]$. $\square$

8.4. **Central value of the Estermann function.**

Proof of Corollary 1.7. The Estermann function D corresponds to D_0 in notation of [Bet16]. For $x \in \mathbb{Q}$, it is the analytic continuation of

$$D(s,x) = \sum_{n\geq 1} \frac{\mathrm{e}(nx)\tau(n)}{n^s},$$

initially defined for $\mathrm{Re}(s) > 1$, evaluated at $s = \frac{1}{2}$. We recall that $\tau(n)$ is the number of divisors of n. We use [Bet16, Lemma 10], noting that the quantity v_{j-1}/v_j corresponds to $T^{j-1}(x)$. Therefore

$$D(\tfrac{1}{2}, x) = \zeta(\tfrac{1}{2})^2 + \sum_{j=1}^{r} \phi_j(T^{j-1}(x)),$$

where

$$\begin{aligned}\phi_j(x) = {}&\tfrac{1}{2}x^{-1/2}\big(\log(1/x) + \gamma_0 - \log(8\pi) - \tfrac{\pi}{2}\big)\\ &+ (-1)^{j-1} i \tfrac{1}{2} x^{-1/2}\big(\log(1/x) + \gamma_0 - \log(8\pi) + \tfrac{\pi}{2}\big) + \zeta\big(\tfrac{1}{2}\big)^2 + \mathcal{E}((-1)^j x),\end{aligned}$$

and $\mathcal{E}$, which corresponds to $\mathcal{E}(0,\cdot)$ in the notation of [Bet16, p. 6900], is bounded and continuous. By comparing the cases $N = 0$ and $N = 1$ of [Bet16, eq. (3.17)], we have

$$\mathcal{E}(x) = \mathcal{E}_1(x) + \sum_{j\in\{1,2\}} \frac{(-1)^j}{j\pi} \Gamma(\tfrac{1}{2}+j)^2 \Big(\frac{x}{2\pi i}\Big)^j \big(D_j(\tfrac{1}{2}, -1/x) + \zeta(\tfrac{1}{2}+j)^2/j\big), \tag{8.12}$$

where $\mathcal{E}_1 \in \mathcal{C}^1([0,1])$ and $D_j(\frac{1}{2}, x) := \sum_{n\geq 1} \tau(n)\mathrm{e}(nx)n^{-\frac{1}{2}-j}$. For $j \geq 1$, the function $D_j(\frac{1}{2}, \cdot)$ belongs to $H^{1/2-\varepsilon}([0,1], \mathbb{C})$ for all $\varepsilon \in (0, 1/2)$. We deduce by taking differences that the right-hand side of (8.12) defines, for all $n \geq 1$, a function in $H^{1/2-\varepsilon}([\frac{1}{n+1}, \frac{1}{n}], \mathbb{C})$. By an argument identical to Lemma 8.3, we also have the bound $\|\phi_j \circ h\|_{(1/2-\varepsilon)} \ll 1$ for all $h \in \mathcal{H}^2$, which validates the hypothesis (2.4) with any $\lambda_0 < \frac{1}{1-2\varepsilon}$.

Note that ϕ_j depends only on the parity of j. We apply Theorem 2.1 with $m = 2$, $d = 2$ and $\alpha_0 = 2 - \varepsilon$. In the estimate (2.6) we evaluate the integrals by means of Lemma A.6. We deduce that there is a constant $\mu \in \mathbb{C}$ such that, letting $\sigma = 1/\pi$ and

$$\chi(t) := \mathbb{E}_Q\Big(\exp\Big\{i\,\mathrm{Re}\Big(t\frac{D(\frac{1}{2}, x) - \mu \log Q}{\sigma(\log Q)^{1/2}(\log\log Q)^{3/2}}\Big)\Big\}\Big), \qquad (t \in \mathbb{C}),$$

we have

$$\chi(t) = \exp\Big\{-\frac{|t|^2}{2} + O\Big(\frac{1}{Q^\delta} + \frac{|t| + |t|^2}{(\log\log Q)^{1-\varepsilon}}\Big)\Big\}. \tag{8.13}$$

We then obtain, by the Berry-Essen inequalities and the bound (8.1) for small frequencies, the statement of Corollary 1.7 up to the value of the expectation. We compute the expectation from the initial object, using the expression [IK04, eq. (3.2)] for Ramanujan sums. For $s, w \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(w) > 2$, we have

$$\sum_{q\geq 1} \frac{1}{q^w} \sum_{\substack{1\leq a\leq q\\(a,q)=1}} D(s, a/q) = \zeta(s)^2 \sum_{q\geq 1} \frac{1}{q^w} \sum_{\ell | q} \mu\Big(\frac{q}{\ell}\Big) \frac{\tau(\ell)}{\ell^{s-1}} \prod_{p^\nu \| \ell} (1 - \tfrac{\nu}{\nu+1} p^{-s}),$$

and so by analytic continuation

$$\sum_{q\geq 1} \frac{1}{q^w} \sum_{\substack{1\leq a\leq q\\(a,q)=1}} D(\tfrac{1}{2}, a/q) = \zeta(\tfrac{1}{2})^2 \zeta(w)^{-1} \zeta(w - 1/2)^2 H(w),$$

where $H(w)$ is analytic and bounded in $\mathrm{Re}(w) \geq 4/3$. We deduce by a standard application of Perron's formula that

$$\mathbb{E}_Q(D(\tfrac{1}{2}, x)) \ll_\varepsilon Q^{-1/2+\varepsilon}. \tag{8.14}$$

On the other hand, by (8.13), we have

$$\chi(t) = 1 + O_\varepsilon(Q^{-\delta} + (\log\log Q)^{-1/2+\varepsilon} t + t^2),$$

whereas exanding the exponential as $\mathrm{e}^{iu} = 1 + iu + O(u^{3/2})$ in the definition of $\chi(t)$, by (8.14), we get

$$\chi(t) = 1 + it\mu(\log Q)^{1/2}(\log\log Q)^{-3/2} + O(Q^{-1/3} + |t|^{3/2}((\log Q)^{1/2} + \mathbb{E}_Q(|D(\tfrac{1}{2}, x)|^{3/2})).$$

Using the trivial bound $\left|D(\frac{1}{2}, x)\right| \ll \sum_{j=1}^{r(x)} a_j(x)^{3/5}$ and Hölder's inequality, we find

$$\begin{aligned}\mathbb{E}_Q(|D(\tfrac{1}{2}, x)|^{3/2}) &\ll \mathbb{E}_Q\Big(\big(\sum_{j=1}^{r(x)} 1\big)^{1/2}\big(\sum_{j=1}^{r(x)} a_j(x)\big)\Big) \\ &\ll (\log Q)^{5/2}\end{aligned}$$

by the bound $r(x) \ll \log(\mathrm{denom}(x) + 1)$ and [YK75, Theorem]. Setting $t = Q^{-\delta/2}$, we obtain

$$\mu(\log Q)^{1/2}(\log\log Q)^{-3/2} = O_\varepsilon((\log\log Q)^{-1/2+\varepsilon})$$

and so $\mu = 0$, by letting $Q \to \infty$. □

8.5. **Dedekind sums.**

Proof of Corollary 1.9. By [Hic77, Theorem 1], we have for $x \in \mathbb{Q} \cap (0, 1)$ the equality

$$s(x) = \delta_x + \frac{1}{12}\sum_{j=1}^{r} \phi_j(T^{j-1}(x)),$$

where $|\delta_x| \leq \frac{5}{12}$ and $\phi_j(x) := (-1)^{j-1}\lfloor 1/x \rfloor$. Note that ϕ_j depends only on the parity of j. Since changing v by an amount $O(1/\log Q)$ does not affect the right-hand side of (1.20), we may replace $s(x)$ by $s(x) - \delta_x$. Let

$$\chi(t) := \mathbb{E}_Q\Big(\exp\Big\{it\frac{s(x) - \delta_x}{\log Q}\Big\}\Big).$$

Then Theorem 2.1 applies with $d = 1$, $m = 2$ and the functions ϕ_j defined above, with $\alpha_0 = 1-\varepsilon$ (note that $\phi|_{h(\mathcal{I})}$ is constant for all $h \in \mathcal{H}^2$). We use the expression (2.5) and evaluate the integral by means of Lemma A.9, obtaining

$$\chi(t) = \exp\Big\{-\frac{|t|}{2\pi} + O\Big(\frac{1}{Q^\delta} + \frac{|t| + |t|^{1-\varepsilon}}{(\log Q)^{1-\varepsilon}}\Big)\Big\},$$

and we conclude again by the Berry-Esseen inequality. □

Appendix A. Asymptotic expansion of oscillatory integrals

In this appendix, we estimate the integral on the right-hand side of (2.5) in the cases needed for our applications. In probablistic terms, we wish to establish that the law with probability distribution function

$$G(y) = \int_0^1 \mathbf{1}(\phi(x) \leq y)\xi(x)\,\mathrm{d}x$$

belongs to the bassin of attraction of some stable law [Lév25], [IL71, Chap. II.6]: this is related to the rate of decay of $G(-y)$ and $1 - G(y)$ as $y \to \infty$. We are looking for effective error terms. There are many ways in which this can be done, for example inserting a Taylor expansion, using Mellin transform [FS09, Appendix B.7] (see also [Lóp08]), or reducing to incomplete

Gamma integrals [IL71]. We will mostly restrict to the case $d = 1$. Since ϕ is real-valued, by taking conjugates, we may add the restriction

$$t > 0.$$

For $\alpha > 0$ and $\mu \in \mathbb{R}$ define $\mathcal{G}_\alpha(\mu, \rho)$ to be the set of piecewise continuous functions $\phi : (0, 1] \to \mathbb{R}$ such that, for some constants $c_1, c_2, c_* \in \mathbb{C}$, and $t_0 > 0$, we have

$$\int_0^1 \mathrm{e}^{it\phi(x)}\xi(x)\,\mathrm{d}x = 1 + c_1 t + c_2 t^2 + c_* t^\alpha |\log t|^\mu + O(|t|^3 + |t|^\alpha |\log t|^{\mu-\rho}) \tag{A.1}$$

for all $t \in (0, t_0)$. Here t_0 may depend on ϕ. Note that if $\alpha < 1$, the term $c_1 t$ in (A.1) is part of the error term, and likewise for $c_2 t^2$ if $\alpha < 2$. Whenever the expansion (A.1) holds for ϕ, we will denote the coefficients by $c_1(\phi)$, $c_2(\phi)$, $c_*(\phi)$ respectively.

Remark. Note that

$$\int_0^1 \mathrm{e}^{it(\phi\circ T)(x)}\xi(x)\,\mathrm{d}x = \int_0^1 \mathrm{e}^{it\phi(x)}\xi(x)\,\mathrm{d}x. \tag{A.2}$$

A.1. **Taylor estimate.**

Lemma A.1. *Assume that for some $\alpha \in (0, 3]$, we have*

$$K := \int_0^1 |\phi(x)|^\alpha\,\mathrm{d}x < \infty.$$

Then $\phi \in \mathcal{G}_\alpha(0, 0)$, i.e.

$$\int_0^1 \mathrm{e}^{it\phi(x)}\xi(x)\,\mathrm{d}x = 1 + c_1 t + c_2 t^2 + O(K t^\alpha)$$

with $c_1 = i \int \phi\xi\,\mathrm{d}\nu$ if $\alpha \geq 1$, and $c_2 = -\frac{1}{2}\int |\phi|^2 \xi\,\mathrm{d}\nu$ if $\alpha \geq 2$. The implied constant is absolute.

Proof. We use the bound $\left|\mathrm{e}^{iu} - \sum_{0\leq k<\alpha} \frac{(iu)^k}{k!}\right| \ll |u|^\alpha$ with $u = t\phi(x)$, and integrate over x. ☐

A.2. **Using properties of the Mellin transform.** One method to obtain the estimate (A.1) is to inspect the polar properties of a Mellin transform. For $x \in (0, 1)$ and $\eta \in [0, 1]$, let

$$\phi_{s,\eta}(x) := \mathbf{1}_{\phi(x)\neq 0} |\phi(x)|^s \mathrm{e}(-\tfrac{s}{4}(1-\eta)\operatorname{sgn}\phi(x)), \qquad \phi_s(x) := \phi_{s,0}(x).$$

Note that for $k \in \mathbb{N}$, $\phi_k(x) = (-i\phi(x))^k$. Define further

$$G_\eta(s) := \int_0^1 \phi_{s,\eta}(x)\xi(x)\,\mathrm{d}x.$$

Lemma A.2. *Let $\alpha, \delta, \rho, \eta_0 > 0$ and $\mu \in \mathbb{R}$. Assume that for some $c > 0$, we have*

$$\int_{\phi(x)\neq 0} (|\phi(x)|^c + |\phi(x)|^{-c})\,\mathrm{d}x < \infty \tag{A.3}$$

and that the functions $G_\eta(s)$ for $\eta \in [0, \eta_0]$, initially defined for $\mathrm{Re}(s) \in (-c, c)$, *can be analytically continued to the set*

$$\{s \in \mathbb{C}, 0 < \mathrm{Re}(s) \leq \alpha + \delta, s \notin [\alpha, \alpha+\delta]\}.$$

Assume further that

$$\sup_{0\leq\eta\leq\eta_0} \int_{\substack{\tau\in\mathbb{R}\} \\ s=\alpha+\delta+i\tau}} |\Gamma(-s)G_\eta(s)|\,\mathrm{d}\tau < \infty,$$

and that there is an open neighborhood $\mathcal{V}$ of $[\alpha, \alpha+\delta]$ for which

$$(\alpha - s)^\mu G_0(s) = \varrho + O(|s-\alpha|^\rho), \qquad s \in \mathcal{V} \smallsetminus [\alpha, \alpha+\delta],\ \mathrm{Re}(s) \leq \alpha + \delta. \tag{A.4}$$

– *If $\alpha \notin \{1, 2\}$, then $\phi \in \mathcal{G}_\alpha(\mu - 1, \rho)$ with coefficients given by*

$$c_* = \varrho\frac{\Gamma(-\alpha)}{\Gamma(\mu)}, \qquad c_1 = -G_0(1) \text{ if } \alpha > 1, \qquad c_2 = \tfrac{1}{2}G_0(2) \text{ if } \alpha > 2.$$

– *If $\alpha = 1$, then $\phi \in \mathcal{G}_1(\mu, \rho)$, with $c_* = -\varrho/\Gamma(\mu+1)$.*

– *If* $\alpha = 2$, *then* $\phi \in \mathcal{G}_2(\mu, \rho)$, *with* $c_* = \frac{1}{2}\varrho/\Gamma(\mu+1)$ *and* $c_1 = -G_0(1)$.

Proof. We write

$$\int_0^1 \mathrm{e}^{it\phi(x)}\xi(x)\,\mathrm{d}x = I_+ + I_- + I_0,$$

where $I_\pm$ corresponds to the part of the integral restricted to $\pm\phi > 0$.

We focus of I_+. For all $\varepsilon > 0$, define

$$I_+(\varepsilon) := \int_{\phi(x)>0} \mathrm{e}^{(-\varepsilon+i)t\phi(x)}\xi(x)\,\mathrm{d}x, \qquad I_-(\varepsilon) := \int_{\phi(x)<0} \mathrm{e}^{(\varepsilon+i)t\phi(x)}\xi(x)\,\mathrm{d}x.$$

By dominated convergence, we have $I_+ := \lim_{\varepsilon\to 0^+} I_+(\varepsilon)$, and similarly for I_-, By the Mellin transform formula for the exponential (see [GR07, eq. 17.43.1]; we require the extension to $\mathrm{Re}(x) > 0$, which is straightforward by the Stirling formula [GR07, eq. 8.327.1]), we obtain

$$I_+(\varepsilon) + I_-(\varepsilon) = \frac{1}{2\pi i}\int_{-c/2-i\infty}^{-c/2+i\infty} \Gamma(-s)G_\eta(s)|1+i\varepsilon|^s t^s\,\mathrm{d}s,$$

where $\eta = \frac{2}{\pi}\arctan\varepsilon$. We move the contour forward to $\mathrm{Re}(s) = \alpha+\delta$,

$$\begin{aligned} I_0 + I_+(\varepsilon) + I_-(\varepsilon) = 1 + R + \frac{1}{2\pi i}\int_{H(\alpha,\alpha+\delta)} \Gamma(-s)G_\eta(s)t^s|1+i\varepsilon|^s\,\mathrm{d}s \\ + \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\alpha+\delta} \Gamma(-s)G_\eta(s)t^s|1+i\varepsilon|^s\,\mathrm{d}s, \end{aligned}$$

where R consists of the contribution of the residues at 1 (if $\alpha > 1$) and 2 (if $\alpha > 2$). Here $H(\alpha, \alpha+\delta)$ is a Hankel contour, going from $\alpha+\delta - i0$ to $\alpha+\delta+i0$ passing around α from the left. The last integral is bounded by the triangle inequality, using our first hypothesis on G_η, which gives

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\alpha+\delta} \Gamma(-s)G_\eta(s)t^s|1+i\varepsilon|^s\,\mathrm{d}s \ll t^{\alpha+\delta}.$$

Passing to the limit $\varepsilon \to 0$, there remains to prove

$$\frac{1}{2\pi i}\int_{H(\alpha,\alpha+\delta)} \Gamma(-s)G_0(s)t^s\,\mathrm{d}s = c_* t^\alpha|\log t|^\mu + O(t^\alpha|\log t|^{\mu-\rho}).$$

This is done by using our second hypothesis along with a standard Hankel contour integration argument; we refer to *e.g.* Corollary II.0.18 of [Ten15] for the details. □

An important special case is the following.

Corollary A.3. *Let* $a \in \mathbb{R} \smallsetminus \{0\}$. *For all* $\beta > 0$ *and* $\lambda > -\beta$, *the function* ϕ *given by*

$$\phi(x) = ax^{-\beta}|\log x|^\lambda$$

satisfies the hypothesis of Lemma A.2 with any fixed $c < 1/\beta$, $\alpha = 1/\beta$, $\mu = \lambda/\beta + 1$, *any fixed* $\rho \in (0,1)$, *and*

$$\varrho = |a|^{1/\beta}\mathrm{e}\Big(\frac{-\operatorname{sgn} a}{4\beta}\Big)\frac{\Gamma(\lambda/\beta+1)}{\beta^{\lambda/\beta+1}\log 2}.$$

Proof. The condition (A.3) is satisfied for any $c < 1/\beta$. The polar behaviour is obtained by analyzing the behaviour of the integral (A.1) near $x = 0$. For $\mathrm{Re}(s) < 1/\beta$, we have by [GR07, 4.272.6]

$$\int_0^1 x^{-\beta s}|\log x|^{\lambda s}\,\mathrm{d}x = \frac{\Gamma(\lambda s+1)}{(1-\beta s)^{\lambda s+1}},$$

and from this, the other conditions of Lemma A.2 and the analytic continuation are easily verified. □

A.3. **Addition.**

Lemma A.4. *For $j \in \{1,2\}$, let $\delta_j(x) = \mathrm{e}^{it\phi_j(x)} - 1$, and $\Delta_j(t) := \int_0^1 (\mathrm{e}^{it\phi_j(x)} - 1)\xi(x)\,\mathrm{d}x$. Then*

$$\int_0^1 \mathrm{e}^{it(\phi_1(x)+\phi_2(x))}\xi(x)\,\mathrm{d}x = 1 + \Delta_1(t) + \Delta_2(t) + O\Big(\int_0^1 |\delta_1(x)\delta_2(x)|\,\mathrm{d}x\Big)$$
$$= 1 + \Delta_1(t) + \Delta_2(t) + O\Big(\prod_{j\in\{1,2\}} |\mathrm{Re}\,\Delta_j(t)|^{1/2}\Big) \tag{A.5}$$

Proof. The first equation is simply the relation $\mathrm{e}^{it(\phi_1(x)+\phi_2(x))} = 1 + \delta_1(x) + \delta_2(x) + \delta_1(x)\delta_2(x)$ integrated over $x \in (0,1)$. The error term is bounded using the Cauchy-Schwarz inequality

$$\Big(\int_0^1 |\delta_1(x)\delta_2(x)|\xi(x)\,\mathrm{d}x\Big)^2 \le \prod_{j\in\{1,2\}} \int_0^1 |\delta_j(x)|^2\xi(x)\,\mathrm{d}x$$

and expanding the square on the right-hand side. □

Corollary A.5. *Let $r \in \mathbb{N}_{>0}$, and for $1 \le j \le r$, let $\alpha_j, \mu_j, \rho_j > 0$ and $\phi_j \in \mathcal{G}_{\alpha_j}(\mu_j, \rho_j)$. Assume that for all $j < r$, we have $t^{\alpha_{j+1}}|\log t|^{\mu_{j+1}} = O(t^{\alpha_j}|\log t|^{\mu_j})$ as $t \to 0$; in other words, the pairs $(\alpha_j, -\mu_j)$ are sorted in decreasing lexicographical order. Assume also that $t^2 = O(t^{\alpha_1}|\log t|^{\mu_1})$. Then*

$$\phi_1 + \cdots + \phi_r \in \mathcal{G}_{\alpha_1}(\mu_1, \rho), \qquad \rho = \begin{cases} \rho_1 & \text{if } \alpha_2 > \alpha_1, \\ \min(\rho_1, \frac{\mu_1-\mu_2}{2}, \frac{\mu_1}{2}) & \text{if } \alpha_2 = \alpha_1 > 2, \\ \min(\rho_1, \frac{\mu_2-\mu_1}{2}) & \text{if } \alpha_2 = \alpha_1 = 2. \end{cases}$$

Moreover,

$$c_1(\phi_1 + \cdots + \phi_r) = c_1(\phi_1) + \cdots + c_1(\phi_r),$$
$$c_*(\phi_1 + \cdots + \phi_r) = c_*(\phi_1).$$

Proof. Use Lemma A.4 iteratively. □

Remark. Note that using this result might induce a slight quantitative loss in the case $\alpha_1 = \alpha_2$, for which we expect $\mu_1 - \mu_2$ (resp. μ_1) instead of $\frac{\mu_1-\mu_2}{2}$ (resp. $\frac{\mu_1}{2}$) on the right-hand side. What is gained at this price is that we are only required to study each ϕ_j separately, which simplifies the analysis.

We also remark that this estimate is useful only when the term c_2t^2 is not relevant in (A.1). In the complementary case, Lemma A.1 can be used, although the ensuing error term will typically be worse than optimal by a factor of $|\log t|$.

It is straightforward to generalize Corollary A.5, affecting to each ϕ_j a different value of the frequency: under the same hypotheses and notations,

$$\int_0^1 \mathrm{e}^{it_1\phi_1(x)+\cdots+it_r\phi_r(x)}\xi(x)\,\mathrm{d}x = 1 + c_1(\phi_1)t_1 + \cdots + c_1(\phi_r)t_r$$
$$+ c_* t_1^{\alpha_1}|\log t_1|^{\mu} + O(t_+^3 + t_+^{\alpha_1}|\log t_+|^{\mu_1-\rho})$$

where c_1, c_* and ρ are as in the conclusion of Corollary A.5, and $t_+ = \max\{t_1, \cdots, t_r\}$.

A.4. **Applications.**

Lemma A.6. *Let $\varepsilon > 0$, $\mathcal{E} : [0,1] \to \mathbb{C}$ be a bounded, continuous function, and*

$$\boldsymbol{\phi}_j(x) := \begin{pmatrix} \frac{1}{2}x^{-1/2}\big(\log(1/x) + \gamma_0 - \log(8\pi) - \frac{\pi}{2}\big) + \zeta(\frac{1}{2})^2 + \mathrm{Re}\,\mathcal{E}((-1)^j x) \\ \frac{(-1)^{j-1}}{2}x^{-1/2}\big(\log(1/x) + \gamma_0 - \log(8\pi) + \frac{\pi}{2}\big) + \mathrm{Im}\,\mathcal{E}((-1)^j x) \end{pmatrix}.$$

Let also $\boldsymbol{u}_j := \binom{1}{(-1)^{j-1}}$. *Then for some vector* $\boldsymbol{\mu} \in \mathbb{R}^2$, *and all* $\boldsymbol{t} \in \mathbb{R}^2$, *we have*

$$\int_0^1 \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(x)+\boldsymbol{\phi}_2(T(x))\rangle}\xi(x)\,\mathrm{d}x$$
$$= 1 + i\langle \boldsymbol{t}, \boldsymbol{\mu}\rangle - \frac{1}{3\log 2}\sum_{j\in\{1,2\}}\langle \boldsymbol{t}, \boldsymbol{u}_j\rangle^2 |\log|\langle \boldsymbol{t}, \boldsymbol{u}_j\rangle||^3 + O_\varepsilon(\|\boldsymbol{t}\|^2|\log\|\boldsymbol{t}\||^{2+\varepsilon})$$

where $\boldsymbol{\mu} \in \mathbb{R}^2$.

Proof. Let $\rho = 1-\varepsilon$. Using Lemma A.3 with $\beta = 1/2$ and $\lambda \in \{0,1\}$, and Lemma A.1, we obtain

$$(x \mapsto \pm\tfrac{1}{2}x^{-1/2}|\log x|) \in \mathcal{G}_2(3,\rho),$$
$$(x \mapsto (\gamma_0 - \log(8\pi) + \tfrac{\pi}{2})x^{-1/2}) \in \mathcal{G}_2(1,\rho),$$
$$(x \mapsto \operatorname{Im}\mathcal{E}(\pm x)) \in \mathcal{G}_3(0,0),$$

as well as $c_*(x \mapsto \pm\frac{1}{2}x^{-1/2}|\log x|) = -\frac{1}{3\log 2}$. From Corollary A.5 and the ensuing remark, and using the property (A.2), we obtain for $j \in \{1,2\}$

$$\int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_j(x)\rangle} - 1)\xi(x)\,\mathrm{d}x = i\langle \boldsymbol{t}, \boldsymbol{\mu}_j\rangle + c_*\langle \boldsymbol{t}, \boldsymbol{u}_j\rangle^2|\log|\langle \boldsymbol{t}, \boldsymbol{u}_j\rangle||^3 + O_\varepsilon(\|\boldsymbol{t}\|^2|\log\|\boldsymbol{t}\||^{2+\varepsilon}),$$

where $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathbb{R}^2$. On the other hand, we have

$$\Delta(\boldsymbol{t}) := \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(x)\rangle} - 1)(\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_2(T(x))\rangle} - 1)\xi(x)\,\mathrm{d}x = \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_2(x)\rangle} - 1)F_x(t)\,\mathrm{d}x,$$

where

$$F_x(\boldsymbol{t}) = \frac{1}{\log 2}\sum_{n\geq 1}\frac{\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(1/(n+x))\rangle} - 1}{(n+x)(n+x+1)}.$$

By a Taylor expansion at order 1, we have $|F_x(\boldsymbol{t})| \ll \|\boldsymbol{t}\|$ uniformly in x, and therefore

$$|\Delta(\boldsymbol{t})| \ll \|\boldsymbol{t}\|^2\int_0^1 \|\boldsymbol{\phi}_2(x)\|\,\mathrm{d}x \ll \|\boldsymbol{t}\|^2.$$

By (A.5), we deduce

$$\int_0^1 \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(x)+\boldsymbol{\phi}_2(T(x))\rangle}\xi(x)\,\mathrm{d}x = 1 + \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_1(x)\rangle} + \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}_2(T(x))\rangle} - 2)\xi(x)\,\mathrm{d}x + O(\|\boldsymbol{t}\|^2),$$

whence the claimed estimate. □

Lemma A.7. *The function* ϕ *given by* $\phi(x) = \lfloor 1/x \rfloor$ *satisfies*

$$\int_0^1 \mathrm{e}^{it\phi(x)}\xi(x)\,\mathrm{d}x = 1 - \tfrac{it}{\log 2}(|\log t| + \gamma_0 - \tfrac{\pi i}{2}) + O_\varepsilon(t^{2-\varepsilon}).$$

Proof. The integral (A.3) converges for all $c < 1$. A quick computation shows that an analytic continuation of $G_\eta(s)$ is given by

$$G_\eta(s) = \frac{\mathrm{e}(-\frac{s}{4}(1-\eta))}{\log 2}\{\zeta(2-s) + H(s)\},$$

where $H(s) = \sum_{n\geq 1} n^s(\log(1+\frac{1}{n(n+2)}) - \frac{1}{n^2})$ is analytic and uniformly bounded in $\operatorname{Re}(s) \leq 2-\varepsilon$. We have

$$\int_{\operatorname{Re}(s)=2-\varepsilon} |\Gamma(-s)G_\eta(s)||\mathrm{d}s| \ll_\varepsilon 1 + \int_0^\infty |\zeta(\varepsilon + i\tau)|\frac{\mathrm{d}\tau}{1+\tau^2} \ll_\varepsilon 1$$

by the Stirling formula. The polar behaviour (A.4) is given by

$$G_0(s) = \frac{\mathrm{e}(-\frac{s}{4})}{\log 2}\{\zeta(2-s) + H(s)\} = \frac{\mathrm{e}(-\frac{s}{4})}{\log 2}\Big\{\frac{1}{1-s} + A + O(s-1)\Big\}$$

for s in a neighborhood of 1, where

$$A=\sum_{n\geq 1}\Big(n\log\Big(1+\frac{1}{n(n+2)}\Big)-\log\Big(1+\frac{1}{n}\Big)\Big)=\int_1^\infty\frac{-\{x\}-1}{x(1+x)}\,\mathrm{d}x=-1.$$

Applying Lemma A.2 with $\delta=1/2$ and $\alpha=1$ yields the claimed result up to $O(t)$. Our more precise statement follows from noting that there is no branch cut along $s\geq 1$ in this case, so that the residue theorem may be used. We obtain

$$\operatorname*{Res}_{s=1}\Gamma(-s)G_0(s)t^s=\tfrac{it}{\log 2}(\gamma_0-\tfrac{\pi i}{2}+\log t),$$

whence the claimed estimate. One could go further, isolating a pole of order 2 at $s=2$, and this would give an error term $O(t^2|\log t|)$. □

Lemma A.8. *Let $\lambda\geq 1/2$. The function ϕ_λ given by $\phi_\lambda(x)=\lfloor 1/x\rfloor^\lambda$ satisfies the following.*
– *If $\lambda=1/2$, then with $c_*=-1/(2\log 2)$, we have*

$$\int_0^1 \mathrm{e}^{it\phi_{1/2}(x)}\xi(x)\,\mathrm{d}x=1+c_1t+c_*t^2|\log t|+O_\varepsilon(t^2|\log t|^\varepsilon). \tag{A.6}$$

– *If $\lambda\notin\{1/2,1\}$, then with $c_*=-\mathrm{e}(-1/(4\lambda))\Gamma(1-1/\lambda)/\log 2$, we have*

$$\int_0^1 \mathrm{e}^{it\phi_\lambda(x)}\xi(x)\,\mathrm{d}x=1+(\mathbf{1}_{\lambda<1})c_1t+c_*t^{1/\lambda}+O_\varepsilon(t^{1/\lambda}|\log t|^{-1+\varepsilon})$$

When $\lambda<1$, we have $c_1=i\int_0^1\phi_\lambda(x)\xi(x)\,\mathrm{d}x$.

Proof. We write $\phi_\lambda(x)=p_\lambda(x)+r_\lambda(x)$, where $p_\lambda(x)=x^{-\lambda}$ and $r_\lambda(x)\ll_\lambda\lfloor 1/x\rfloor^{\lambda-1}$.

We consider first the case $\lambda>1/2$, $\lambda\neq 1$. Then we have $r_\lambda\in\mathcal{G}_{\min(2,1/(\lambda-1))}(0,0)$ by Lemma A.1, while by Lemma A.3, we have $p_\lambda\in\mathcal{G}_{1/\lambda}(0,1-\varepsilon)$. We deduce, by Corollary A.5, that $\phi_\lambda\in\mathcal{G}_{1/\lambda}(0,1-\varepsilon)$, and this yields the second and third cases.

If $\lambda=1/2$, then Lemma A.3 implies $p_{1/2}\in\mathcal{G}_2(1,1-\varepsilon)$, and by Lemma A.1, for some $c\in i\mathbb{R}$, we have

$$\int_0^1(\mathrm{e}^{itr_{1/2}(x)}-1)\xi(x)\,\mathrm{d}x=ct+O(t^2)$$

On the other hand, since $\Big|(\mathrm{e}^{itp_{1/2}(x)}-1)(\mathrm{e}^{itr_{1/2}(x)}-1)\Big|\ll t^2\Big|p_{1/2}(x)r_{1/2}(x)\Big|\ll t^2$, we get

$$\int_0^1(\mathrm{e}^{itp_{1/2}(x)}-1)(\mathrm{e}^{itr_{1/2}(x)}-1)\xi(x)\,\mathrm{d}x=O(t^2).$$

By (A.5), we conclude (A.6) as claimed. □

Lemma A.9. *We have*

$$\int_0^1 \mathrm{e}^{it(\lfloor 1/x\rfloor-\lfloor 1/T(x)\rfloor)}\xi(x)\,\mathrm{d}x=-\frac{\pi}{\log 2}t+O_\varepsilon(t^{2-\varepsilon}).$$

Proof. We consider

$$\begin{aligned}\Delta(t)&:=\int_0^1(\mathrm{e}^{-it\lfloor 1/T(x)\rfloor}-1)(\mathrm{e}^{it\lfloor 1/x\rfloor}-1)\xi(x)\,\mathrm{d}x\\&=\int_0^1(\mathrm{e}^{-it\lfloor 1/x\rfloor}-1)F_x(t)\,\mathrm{d}x,\end{aligned}$$

with $F_x(t)=\frac{1}{\log 2}\sum_{n\geq 1}\frac{\mathrm{e}^{itn}-1}{(n+x)(n+1+x)}$. Since $|\mathrm{e}^{iu}-1|\ll|u|^{1-\varepsilon}$ for all $u\in\mathbb{R}$, we find

$$F_x(t)\ll t^{1-\varepsilon}\sum_{n\geq 1}\frac{1}{n^{1+\varepsilon}}\ll_\varepsilon t^{1-\varepsilon}.$$

Similarly,

$$\int_0^1\Big|\mathrm{e}^{-it\lfloor 1/x\rfloor}-1\Big|\,\mathrm{d}x\ll t^{1-\varepsilon}\int_0^1 x^{-1+\varepsilon}\,\mathrm{d}x\ll_\varepsilon t^{1-\varepsilon}.$$

We thus obtain $\Delta(t) = O_\varepsilon(t^{2-\varepsilon})$. Using (A.5) and (A.2), we deduce

$$\int_0^1 \mathrm{e}^{it(\lfloor 1/x\rfloor - \lfloor 1/T(x)\rfloor)}\xi(x)\,\mathrm{d}x = 1 + 2\operatorname{Re} I(t) + O_\varepsilon(t^{2-\varepsilon}),$$

where $I(t) = \int_0^1 (\mathrm{e}^{it\lfloor 1/x\rfloor} - 1)\xi(x)\,\mathrm{d}x$. Lemma A.7 allows us to conclude. $\square$

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